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Typical and Atypical Mathematics Learning: What Do We Learn From Recent Studies?

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Typical and Atypical Mathematics Learning: What Do We Learn From Recent Studies?

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Abstract

Poor mathematical abilities have a substantial societal impact. This special issue includes contributions discussing learning of mathematics that have impact on educational policy or elementary school practice. All papers explore and illustrate recent studies and available literature in the field of understanding typical and atypical mathematics learning. They reflect on mechanisms of conceptual development, models of typical and atypical learning, individual quantitative and qualitative differences, strengths and weaknesses, factors that appear promising and positively influencing the learning process of students' learning of mathematics, and on interventions that aim to improve mathematics performance even in very young children.

Keywords:

Mathematics, Models, Differences, Interventions

Introduction

Mathematical abilities are needed in all kinds of everyday situations (e.g., understanding corona virus statistics, calculating prices, estimating amounts and many more). Although mathematical abilities seem to be learned quite effortlessly in most people, some children have persisting problems with acquiring and/or applying these abilities. This is problematic because mathematical ability is an important predictor of later academic achievement, and since poor mathematical skills may lead to decreased perceived competence and increased emotional and behavioral disengagement. In addition, poor mathematical abilities often result in gaining employment in low paid professions during adulthood and have negative consequences for mental health (Ritchie & Bates, 2013). Mathematical disabilities thus have a substantial societal impact.

Up till now research in relation to typical and atypical mathematics learning and to individuals with mathematical learning disabilities remains an underrepresented area of research. As a result of the limited studies, it remains unclear if and on which components of mathematical ability individuals show strengths or difficulties, what predicts typical and atypical achievement and how mathematical underachievement can be detected, screened, understood, and addressed.

This special issue brings together studies and reviews about mathematical abilities that may have an impact on educational policy and/or educational school practice. In this way the issue aims to gain knowledge about the development of mathematical abilities and to map the typical and atypical development of mathematics learning. This may advance the understanding of the human abilities to perceive, represent, learn, and manipulate mathematical information.

The various contributions in this special issue

First, papers explored models of typical and atypical learning. Allen and Dowker (2022) studied the relationships between visuo-spatial working memory and different types of arithmetic in 39 children in Year 2 (6 to 7 years) and Year 4 (8 to 9 years). In addition Desoete and Baten (2022) studied the prediction of propensity (intelligence, motivation and subjective wellbeing), opportunity (years of experience of teachers and the number of hours of mathematical instruction children receive) and antecedent (gender, parental aspirations, birth order and birth weight) factors to predict mathematics in 408 children from grade 4 or 5 using the Opportunity Propensity framework (Byrnes, 2020) to prevent overestimation of the importance and unique explained variance of predicting factors. Finally, Kroesbergen (2022) proposed a multidimensional framework in which children should be regarded as individuals with unique profiles of strengths and weaknesses.

Second, papers explored mathematics learning in different age groups. Olkun (2022) reviewed studies on learning numbers with the framework of "number sense". Hartmann (2022) assessed the counting skills of 107 preschoolers (mean age 57.61 months) to focus on the conceptual understanding of "zero" as number word for an empty set emerges. Şenol (2022) examined 132 children aged 60–72 months with normal development who attend preschool education on the relationship between the academic competencies and their social information processing processes. Finally van Dijck, Abrahamse, Kesteloot, Willems and Fias (2022) focused on 438 first year bachelor students in Psychology and observed that high levels of motivation could alleviate the negative impact of statistics anxiety on statistical performance, especially when controlling for general learning abilities.

Third, contributions focused on individual differences and on the assessment of children with and without mathematical learning disabilities or dyscalculia. Martin, Mraz, and Polly (2022) studied the perceptions of 65 teachers and their use of formative assessment in mathematics, as formative assessment can be seen as a high-leverage instructional practice that has potential to support all learners. Mononen, Niemivirta and Korhonen (2022) investigated in 206 participants numeracy, cognitive, and language skills in grade 1 and arithmetic fluency and curriculum-based mathematics in grade 3. Korkmaz and Temur (2022) used electrophysiological measures in a pilot study on third and fourth graders with learning difficulties in mathematics. Finally, Lewis (2022) described how the current assessment of dyscalculia resulted in an over-representation of students of color, non-native speakers, and students from low SES backgrounds. To address this problem, they set up two studies assessing 470 grade 6-8 students and three students who demonstrated high levels of unconventional understandings.

Finally, this special issue also included interventions focusing on mathematical ability. These contributions suggest some good practices focusing on mathematical ability (or subcomponents). Diago (2022) focused on the improvement of counting skills in 14 children without special needs aged between 3 years 5 months and 4 years 4 months using special designed tasks. Akıncı-Cosgun (2022) examined the effect of a training program for 21 children between 48–65-month-old on early mathematics ability and mother-child relationship at home. Urton, Grünke and Boon (2022) studied the effect of multisensory mathematics instruction integrating a touch points strategy, performance feedback, reward system, and a reinforcing game into an instructional package on the subtraction performance of 4 children at-risk for learning disabilities aged between 6 and 7 years. Lee and Hwang (2022) explored the solving of word problems, helping 7 students to recognize multiple relationships within the context of specific problems and real-world related applications. Korkmaz and Temur (2022) examined in 4 third and fourth graders with learning difficulties in mathematics if music support enhanced calculation skills. Herzog and Casale (2022) studied the effectivity of a computer-based mathematics intervention on 11 children with and without emotional and behavioral difficulties in grades 3 and 4, pointing to different effectiveness for children with and without such additional difficulties. Finally, Alqahtani and colleagues (2022) focused on an intervention on the representational models of fractions of 46 pre-service elementary teachers.

Some preliminary observations

The papers included in this special issue demonstrate that mathematical abilities depend on multiple factors such as domain-specific knowledge and skills (e.g., magnitude processing, counting, calculation understanding 'zero', models about fractions), domain-general cognitive skills (e.g., visuospatial working memory, intelligence, non-verbal reasoning, rapid naming) and non-cognitive factors (e.g., social information processing, affect, motivation, math anxiety) that also interact with one another.

In addition, different cut-off scores (≤ 25th percentile, ≤ 10th percentile or between 11-25th percentile) and different mathematics measures (arithmetic fluency or curriculum-based measures) seem to lead to different early domain-specific (symbolic numerical magnitude processing, verbal counting) and domain-general (nonverbal reasoning, rapid automatized naming, working memory) predictors.

In young children Şenol (2022) revealed that academic skills (numeracy, early literacy, thinking skills, and comprehension) and achievement (social-emotional competence, approaches to learning, and communication) cannot be studied as isolated phenomena, since they are related to skills to understand cues and decide about responses as subdimensions of the social information processing model.

In elementary schoolchildren, Allen and Dowker (2022) revealed a relationship between visuo-spatial working memory and verbal oral and mental written arithmetic but not with derived fact strategy use. Desoet and Baten (2022) added that intelligence was a significant predictor for math fluency and calculation accuracy, whereas positive affect influenced math fluency negative affect predicted calculation accuracy.

In older participants van Dijck and colleagues (2022) observed that high levels of motivation could alleviate the negative impact of statistics anxiety on statistical performance, especially when controlling for general learning abilities.

Finally, Alqahtani (2022) described that how teachers interpret and express fractions also might critically influence their teaching and their students' fraction knowledge, pointing to the importance of teacher training programs.

When summarizing, there are some findings on the assessment of mathematical learning (dis)abilities. Korkmaz and Temur (2022) made us reflect on the value of electrophysiological measures to assess the effect of interventions. Martin and colleagues (2022) pointed to the value of formative assessment to identify gaps and strengths of the learner, but they also revealed barriers in elementary school teachers using formative assessment. Lewis (2022) pointed to the potential of designing screeners based on the characteristics identified in adults with dyscalculia and Kroesbergen (2022) proposed to look for unique profiles of strengths and weaknesses.

Finally, this special issue also provides recommendations for interventions. Diago (2022) showed how young children can learn to master the counting principles in 16 sessions of 20 minutes, although some of the principles remain more difficult than other principles. Akıncı-Cosgun (2022) revealed that children could learn to recognized numbers and shapes and to know the total number of objects displayed in the group, while parents described better relationships and quality time with their children and increased use of mathematics in daily life. Lee and Hwang (2022) described how the retrospective analysis of class episodes offered insight into learning opportunities to support students in exploring mathematical structure and relationships while discussing and debating the word problem context. Urton and colleagues (2022) revealed that their intervention was effective to very effective to enhance the ability of students to solve subtraction problems. Students' performance improved during the course of the intervention as they learned and practiced the touch points strategy. Alqahtani and colleagues (2022) revealed the value of a measuring perspective to support pre-service teachers to shift from procedural strategies such as symbolic manipulation to more conceptual strategies to identify and represent fractions. Although most of these intervention studies are based on a very limited number of participants, most of them see to have some positive outcome. However, not every intervention was equally successful. Herzog and Casale (2022) revealed that the effectiveness of mathematics interventions might not be generalizable for children with comorbid emotional and behavioral problems, stressing the need for additional studies to address typical and atypical mathematics learning with and without comorbid disorders.

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Spatial Working Memory Counts: Evidence for a Specific Association Between Visuo-spatial Working Memory and Arithmetic in Children

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Abstract

We examined the role of visuo-spatial working memory in different types of arithmetic ability in children. Previous research had suggested that arithmetic is not a single entity (Dowker, 2005, 2015), and also that visuo-spatial working memory is specifically involved in mathematical cognition (McKenzie et al., 2003) There has, however, been little research on the relationships between visuo-spatial working memory and different types of arithmetic. We tested 39 children in Year 2 (6 to 7 years) and Year 4 (8 to 9 years), taking measures of written arithmetic, mental oral arithmetic, and derived fact strategy use (the ability to derive unknown arithmetical facts from known facts, by using arithmetical principles). We also measured visuo-spatial working memory, verbal comprehension, and spelling ability. We investigated the relationships between visuo-spatial working memory and our three arithmetic measures, as well as spelling and verbal comprehension, to test whether these effects were specific to mathematical abilities. We found that visuo-spatial working memory was specifically associated with both verbal oral and mental written arithmetic, and but not with spelling or derived fact strategy use.

Keywords:

Arithmetic Ability, Working Memory, Visuo-Spatial, Mathematics

Introduction

Arithmetic is important in many aspects of our lives. It is important in daily practical activities, such as finding the right change for the bus, estimating how long a journey will take, or comparing different special offers in the supermarket. Basic numeracy is also very important in obtaining and keeping a wide variety of jobs, and low numeracy has many negative social and economic consequences to the individual and to society (Gross et al., 2009; Parsons & Bynner, 2005; Rodgers et al., 2019).

In the light of this prevalence, mathematical instruction in school has potential effects reaching far beyond the classroom, particularly with regard to arithmetic procedures, and so it is crucial to lay the educational foundations of these abilities in the best way possible, and to recognize early when children show characteristics that may contribute to difficulties in learning arithmetic and may indicate a need for interventio. In order to achieve this it is important to understand how different cognitive functions contribute to mathematical ability in children. Furthermore, insight into these relationships can increase our theoretical understanding of the nature and development of arithmetic.

The Multi-Component Nature of Arithmetic

Research suggests that arithmetic is not a single entity; it is made up of multiple components (Dowker, 2005; Jordan, Mulhern & Wylie, 2009). It is important to examine the relationship between these components and the cognitive processes which subserve their development to further our understanding of mathematical cognition. It is likely that multiple processes subserve arithmetical development. These may be used differentially depending on the problem (large vs small numbers, adding vs subtracting), and, crucially, depending on the individual child. In fact, considering the finding that within an average British school class of 11 year olds, there is usually a 7 year range in mathematical ability (Askew, Hodgen, Hossain, & Bretscher, 2010), it seems all the more likely that the processes facilitating arithmetical development are not consistent for all children. Thus, understanding mathematical cognition is important from both a practical perspective, to help children succeed at mathematics, as well as a theoretical one, to further our understanding of the processes that subserve arithmetical development, either independently or in combination. In this study, therefore, we are examining performance in two different types of standardized tests of arithmetic: the WISC Arithmetic subtest, which mainly measures arithmetical reasoning in the context of word-problem solving, and the British Abilities Scales Basic Number Skills Test, which mainly measures written calculation. Note that the most recent, third, edition of the latter test includes a significantly larger element of word problem solving. We chose to use the second edition as a purer measure of written arithmetic,

Derived Fact Strategies

One crucial component of arithmetic is the use of derived fact strategies (Baroody, Ginsburg & Waxman, 1983; Canobi, 2005; Canobi, Reeve & Pattison, 1998; Dowker, 2009, 2014; Gilmore & Papadatou-Papastou, 2009; Godau et al., 2014; Jordan et al., 2009; Robinson et al., 2006; Torbeyns et al., 2009). This involves the ability to derive unknown arithmetical facts from known facts, by using arithmetical principles such as commutativity (if 56 + 31 = 87, then 31 + 56 must also be 87) and the addition/ subtraction inverse principle (if 56 + 31 = 87, then 87 – 56 must be 31). The ability to use derived fact strategies is important both as an indicator of children's understanding of arithmetical principles and relationships, and as a basis for going beyond existing knowledge in performing unfamiliar calculations.

The Role of Working Memory in Arithmetic, and Derived Fact Strategies

While there are numerous cognitive processes that have been proposed to be important to arithmetic, one that has been found in some previous studies to be particularly relevant, and which we have chosen to study in this project, is working memory (Bull & Scerif, 2001; Jarvis & Gathercole, 2003). Working memory is widely accepted to refer to the processes by which information is actively held on-line (Baddeley & Hitch, 1974) and to include both phonological and visual-spatial components. Working memory has been implicated as being used differentially in mathematical processing in children of different ages (McKenzie, Bull, & Gray, 2003; Palmer, 2000). McKenzie et al. (2003) showed that younger children (6-7 years) were on the whole unaffected by verbal interference when working out a mental calculation problem presented verbally, suggesting that the phonological loop was not being used in children at this age. Furthermore, the same children were severely impaired when interference was given in the visuo-spatial modality, suggesting that younger children use visuo-spatial strategies in mental arithmetic. Older children (8-9 year olds) tested in the same experiment were equally impaired by both phonological and visuo-spatial interference, though not to the same extent as the younger children. This suggests that older children use both components of working memory in arithmetic processing. Palmer (2000) suggests that this switch in strategy use accompanies the maturation of the central executive, believed to be involved in switching strategies (Baddeley, 1996).

It would be ideally desirable to investigate the role of all possible components of working memory in mathematical cognition; but given the constraints of testing children within a limited time-scale, a measure of visuo-spatial working memory seemed appropriate to investigate, as it has been previously shown to be important in both older and younger children. Moreover, arithmetic may be more specifically related to visuo-spatial working memory rather than phonological working memory through the use of the internal 'mental number line' (Dehaene, 2011), which is thought to be a spatial mechanism subserving arithmetic. We included a measure of spelling in our study, to investigate whether the role of visual-spatial working memory was indeed specific to arithmetic. With some exceptions, (e.g. Simmons, Willis & Adams, 2012; Szucs et al., 2014) most studies of working memory and arithmetic have not looked at how working memory relates to different types of arithmetic. In particular, to our knowledge, none have looked directly at the extent to which working memory influences derived fact strategy use.

Attitudes to Arithmetic

Arithmetic depends not only on cognitive processes, but also on emotional factors. There is much evidence (OECD, 2015) that attitudes to mathematics have an important effect on performance. Most studies indicate that primary school children have relatively positive attitudes to arithmetic (e.g. Dowker, Bennett & Smith, 2012; Krinzinger, Kaufmann & Willmes, 2009; Sorvo et al., 2017), but that they often deteriorate later on. However, mathematics anxiety is already a problem for some children in the early years of primary school (Petronzi et al., 2019). These studies also suggest that, whereas in older children and adults, the most crucial attitude predictor of performance is mathematics anxiety, in younger children, it seems to be self-rating Therefore we included brief measures of liking mathematics and of self-rating in arithmetic. We predicted that self-rating in particular would predict performance in standard arithmetic tests, but not in derived fact strategy use, as children's self-ratings may be more associated with tasks resembling typical school tests.

Putting It All Together: The Present Study

This study aimed to investigate the relationship between a selection of domain-general and domain-specific cognitive functions and mathematical ability in children over developmental time. We tested children in Years 2 and 4 of the British schooling system (6-7 year olds and 8-9 year olds respectively). As well as several measures of both written and mental mathematical ability including a measure of derived fact strategy use, measures of visuo-spatial working memory were included. Measures of verbal comprehension and spelling ability were also used to act as proxies for other non-mathematical academic abilities, in order to test how specific any relationships between cognitive and mathematic abilities were. Finally, measures of perceived mathematical ability and attitudes towards mathematics were also recorded, to investigate any effects that these may have on mathematical ability.

Our hypotheses were:

1. *Older children would be better at all arithmetical measures than younger children. However, as standard scores are adjusted for age, they should not have higher standard scores.*
2. *There would be a significant correlation between the two standardized arithmetic tests.*
3. *Derived fact strategy use would be strongly related to other measures of arithmetic, and would correlate more strongly with the written arithmetic test than the mental word problem solving test, as previously found by Dowker (2009).*
4. *Visuo-spatial working memory would be a strong predictor of all arithmetical measures. It would predict mental word problem solving (WISC Arithmetic) and derived fact strategy use more than written calculation (BAS Arithmetic).*
5. *Visuo-spatial working memory would not be a strong predictor of spelling.*
6. *Arithmetical self-rating would predict mental word problem solving (WISC Arithmetic) and written calculation (BAS Arithmetic) but not derived fact strategy use.*

Method

Participants

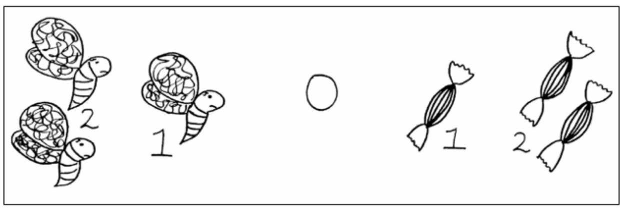
Participants were recruited from two non-selective state primary schools in the Oxford area. The participants consisted of 39 (19 female, 20 male) children in Years 2 and 4 (ages 6-7, and 8-9, respectively). The Year 2 group consisted of 21 children (6 female, 15 male) aged between 6.4 and 7.5 (*M* = 6.9, *SD* = 0.3), and the Year 4 group consisted of 18 children (13 female, 5 male) aged between 8.4 and 9.3 (*M* = 8.10, *SD* = 0.3).

Subtests

Each child completed 7 subtests in total, giving measures of: (a) attitudes towards mathematics and perceived arithmetic ability, (b) arithmetical reasoning and word problem solving (WISC Arithmetic), (c) written calculation (British Abilities Scales Basic Number Skills), (d) use of derived fact strategies, (e) visuo-spatial working memory capacity, (f) verbal comprehension (WISC Comprehension) and (g) spelling ability (British Abilities Scales Spelling).

(a) Attitudes towards mathematics and perceived mathematic ability were assessed using a pictorial rating scale, from Thomas and Dowker (2000) (see Fig. 1).

Figure 1
Pictorial scale used to measure attitudes to mathematics and perceived mathematic ability. From Thomas & Dowker (2000).



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Children were shown the scale, and asked to point to the picture representing how much they liked maths, ranging from two sweets (like maths very much) to two wasps (dislike maths very much) with a neutral circle in the middle (neither like nor dislike maths). Children were then asked to use the scale to show how good they thought they were at maths, ranging from two sweets (very good at maths) to two wasps (not very good at maths at all) with a neutral circle in the middle (neither good nor bad at maths).

Responses were recorded by the examiner onto paper.

(b) The Arithmetic subtest of the Wechsler Intelligence Scale for Children – Fourth Edition (Wechsler, 2004) was administered. Children were given a series of verbally presented arithmetic problems to solve mentally.

(c) Written calculation ability was assessed using the Number Skills subtest of the BAS II (BAS Arithmetic).

(d) Use of derived fact strategies was tested with Dowker et al.'s (2005) Derived Fact Strategy Addition test. Children were first assessed with 20 mental addition questions, which became progressively more difficult, from 4+5 to 235+349 (see Appendix A). Questions were presented visually and read aloud by the examiner, and responses were verbal. Testing continued until the child gave an incorrect response to six items in a row, or completed the test. From this, they were assigned to one of 5 sets for the Derived Fact Strategies test. Each set consisted of 28 pairs of addition questions, designed to test the child's use of six arithmetic principles:

1. *The identity principle (if 8+2 = 10, then 8+2 = 10)*
2. *The commutativity principle (if 8+2=10, then 2+8 = 10)*
3. *The N+1 principle (if 8+2 = 10, then 8+3 = 11)*
4. *The N-1 principle (if 8+2=10, then 8+1=9)*
5. *The N+2 principle (if 8+2=10, then 8+4=12)*
6. *The addition/subtraction inverse principle (if 8+2=10, then 10-2=8)*

The answer to the first question of each pair was given to the child, along with the second question of the pair, which could be solved relatively quickly if the child used the answer to the first question plus the principle being tested. The questions were designed to be slightly too difficult for the child to solve unaided. For example, if the pair of questions was: [349 + 234 = 583, 349 + 236 =], the answer 585 could be given if the child used the answer to the first question of the pair, and the N+2 principle. The children received three addition problems per principle, as well as three addition problems preceded by numerically unrelated problems as controls. If any answers were ambiguous, a fourth question was given for that principle.

(e) Visuo-spatial working memory (VSWM) capacity was tested using the DotMatrix subtest of the Automated Working Memory Assessment (AWMA; Alloway, 2008). This was administered on a laptop screen. Children were asked to recall the location of a series of red dots on a white grid. The dots appeared one at a time, and the sequences of dots became increasingly longer until the child was unable to report their correct locations in the correct order. The sequence length began at 1 dot and increased to a maximum of 9 dots. The child indicated where the dots had been by pointing to locations on the blank grid after each sequence of dots, and the examiner used the keyboard to report whether the child had been correct or incorrect. The test automatically stopped once a certain number of incorrect answers had been recorded.

(f) Verbal comprehension was assessed using the Comprehension subtest of the WISC-III (Wechsler, 2004).

(g) Spelling ability was assessed using the Spelling subtest of the British Ability Scales Second Edition (BAS II; Elliott, Smith, & McCulloch, 1996).

Procedure

Written consent was obtained from a parent or guardian for each child to be included in the study. In addition to this, children were informed that they could choose to stop participating in the study at any time. Before beginning the study, approval was sought and granted from the Oxford University Central University Research Ethics Committee (CUREC).

All children were tested individually in a quiet area of their school. In order to control for practice and fatigue effects, the order in which the components of the study were administered was randomised for each child, using the online software Research Randomizer 4.0 (Urbaniak & Plous, 2013). At the start of their test session, each child was presented with an image of a trophy cabinet printed onto A5 paper (see Appendix B). They were told that they would be able to add one sticker to their cabinet upon completing each subtest, and that they would be able to take it home once the session was over.

The subtests were then administered in a randomised order, with children resting for approximately two minutes in between each subtest.

Results

The data were analysed using SPSS Statistics.

Standard/ Scaled Scores on Standardized Tests

The mean standard score for Working Memory was 119.13 (sd = 13.3). The mean scaled score for the WISC Arithmetic subtest was 13.36 (sd = 2.88) and the mean scaled score for the WISC Comprehension subtest was 12.82 (*sd* = 3.84) The mean scaled score for the British Ability Scales Basic Number Skills subtest was 121.28 (*sd* = 14.14) and the mean scaled score for the British Ability Scales Spelling subtest was 117.38 (*sd* = 14.14).

Significant Differences between Genders and Year Groups

A two-way multivariate analysis of covariance (MANCOVA; α = .05) was conducted with Year Group and Gender as the grouping factors and BAS Spelling Raw Score, BAS Spelling Standard Score, BAS Arithmetic Raw Score, BAS Arithmetic Standard Score,

WISC Arithmetic Scaled Score, WISC Arithmetic Raw Score, WISC Comprehension Scaled Score, WISC Comprehension Scaled Score, VSWM Raw Score, and VSWM Standard Score as the dependent variables. Age (in months) was included as a covariate.

It was found that, after controlling for Age, Year Group still had a significant main effect on VSWM Raw Score, $F(1, 34) = 4.90$, $p = .034$, $\eta^2 = .13$; and BAS Spelling Raw Score, $F(1, 35) = 6.70$, $p = .014$, $\eta^2 = .16$.

For each of these dependent variables on which Year Group had a significant main effect, scores were significantly higher for children in the Year 4 group than the Year 2 group (see Table 1.; see Figure 2.).

Figure 2

Comparison of significantly different mean scores for Year 2 and Year 4 when controlling for Age. Error bars show the Standard Deviation.

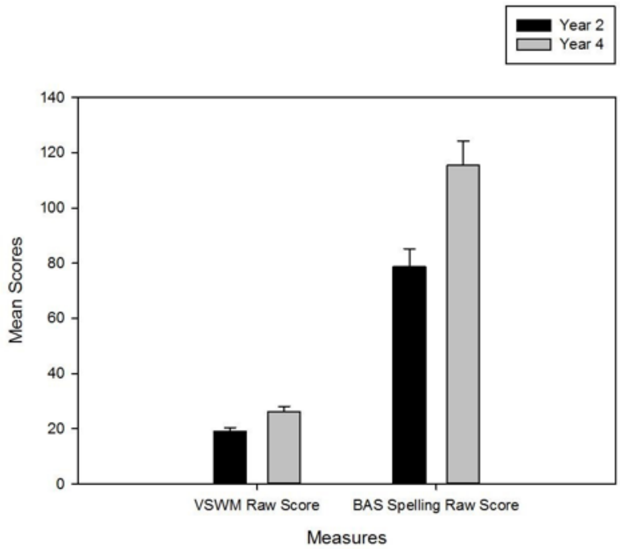

Table 1

Comparison of significantly different mean scores for Year 2 and Year 4 when controlling for Age.

	Year 2		Year 4	
Measure	Mean	*SD*	Mean	*SD*
BAS Spelling Raw Score	83.22	17.69	108.31	10.85
Raw Working Memory	19.61	3.68	26.19	3.33

Gender had a significant main effect on BAS Arithmetic Raw Score, $F(1, 35) = 5.34$, $p = .027$, $\eta^2 = .14$. Furthermore, there was also a significant main effect of Gender on BAS Arithmetic Standard Score, $F(1, 35) = 4.77$, $p = .036$, $\eta^2 = .12$. A comparison of means showed that BAS Arithmetic Raw and Standard Scores were significantly higher for males than for females (see Table 2; see Figure 3.). There were no significant interactions between Year Group and Gender.

Figure 3

Comparison of significantly different mean scores between Genders when controlling for Age. Error bars show the Standard Deviation.

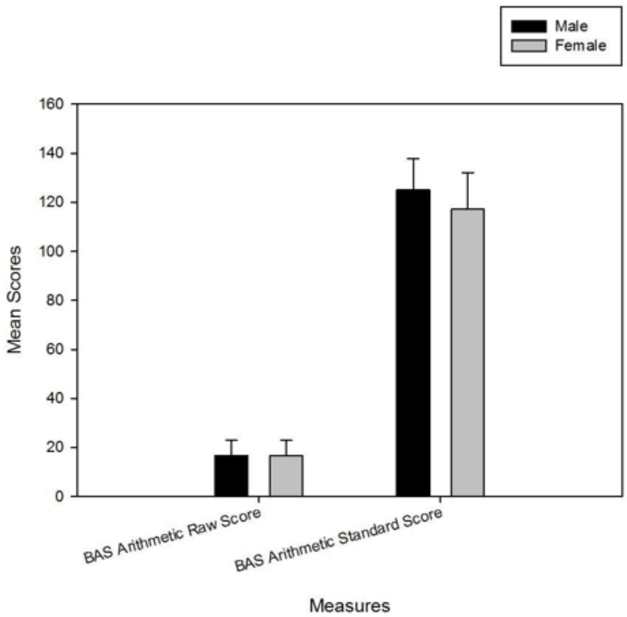

Table 2

Comparison of significantly different mean scores between Genders when controlling for Age.

	Year 2		Year 4	
Measure	Mean	*SD*	Mean	*SD*
BAS Spelling Raw Score	83.22	17.69	108.31	10.85
VSWM Raw Score	19.61	3.68	26.19	3.33

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Derived Fact Strategy Use

Of the 23 Year 2 children, 2 were assigned to Set 2, 13 to Set 3, 4 to Set 4 and 4 to Set 5. All 16 of the Year 4 children were assigned to Set 5.

The mean number of types of derived fact strategy used (out of a potential 6) was 4.1 (*sd* = 1.45), In Year 2, it was 3.7 (*sd* = 1.52) and in Year 4, it was 4.69 (s..d. 1.14). As found in other studies (e.g. Dowker, 2014), there were wide differences in frequency of different strategy types. All children used Identity and nearly all (36 out of 39) used Commutativity.

30 out of 39 (16 out of 23 in Year 2 and 14 out of 16 in Year 4) used Addend + 1. 28 out of 39 (13 out of 23 in Year 2 and 15 out of 16 in Year 4) used Addend – 1. 21 out of 39 (10 out of 23 in Year 2 and 11 out of 16 in Year 4) used Addend + 2.

By contrast with all the above, only 6 out of 39 used Inverse (two out of 23 in Year 2 amd four out of 16 in Year 4.. This was even less common than the production of correct answers to control problems (produced to at least one problem by 9 out of 39 children,)

Correlations between Standardised Measures and Derived Fact Strategy Set, Age, and Number of Strategies Used

Pearson Correlation Coefficients were computed between the standardised measures (VSWM Standard Score, WISC Arithmetic Scaled Score, WISC Comprehension Scaled Score, BAS Arithmetic Standard Score and BAS Spelling Standard Score) and Derived Fact Strategy Set, Age, and Number of Strategies Used. The results are summarised in the correlational matrix below (see Table 3).

Predictors of Number of Strategies Used

A simultaneous entry multiple linear regression was conducted to test the best predictors for the Number of Strategies Used. The predictor variables were Age, BAS Arithmetic Standard Score, BAS Arithmetic Standard Score, Derived Fact Strategy Set and VSWM Standard Score. Number of Strategies Used was the dependent variable. The overall regression was significant, with the four predictor variables explaining 37% of the variance in the Number of Strategies Used, adjusted R^2 = .37, F(5, 33) = 5.42, p = .001. Derived Fact Strategy Set was a significant predictor of Number of Fact Strategies Used, β = .75, t(33) = 3.14, p = .004; and so was BAS Arithmetic Standard Score, β = .37, t(33) = 2.12, p = .042. The remaining predictor variables were not significant at predicting Derived Fact Strategy Set. Thus, BAS Spelling Standard Score, β = -.29, t(33) = -1.82, p = .078; and VSWM Standard Score, β = -.27, t(33) = -1.47, p = .151; and Age, β = .22, t(33) = -0.91, p = .368, did not predict Derived Fact Strategy Set.

Another regression analysis tested how well a different set of predictor variables predicted Number of Strategies Used. The predictor variables were Age, WISC Arithmetic Scaled Score, WISC Comprehension Scaled Score, VSWM Standard Score, and

Derived Fact Strategy Set. Number of Strategies Used remained the dependent variable. The overall regression was significant, with the five predictor variables explaining 26% of the variance in the Number of Strategies Used, adjusted R^2 = .26, F(5, 33)

Table 3

Correlations between standardised measures and Age, Derived Fact Strategies Set, and Number of Strategies Used.

	Measures	1	2	3	4	5	6	7	8
1	VSWM Standard Score	-	.38*	.56**	-0.11	.60**	.45**	0.27	0.21
2	WISC Comprehension Scaled Score		-	0.27	-.33*	0.19	0.31	0.27	.38*
3	WISC Arithmetic Scaled Score			-	0.15	.63**	.46**	0.28	0.21
4	BAS Spelling Standard Score				-	0.17	-0.07	-0.16	-.42*
5	BAS Arithmetic Standard Score					-	0.24	.35*	-0.08
6	Derived Fact Strategy Set						-	.57**	.73**
7	Number of Strategies Used							-	.36*
8	Age								-

* p < .05 ; ** p < .01

= 3.61, p = .010. Of the five predictor variables, Derived Fact Strategy Set was the only significant predictor of Number of Fact Strategies Used, β = .67, $t(33)$ = 2.87, p = .007. None of the other variables was a significant predictor.

Predictors of BAS Arithmetic Standard Score

A simultaneous-entry multiple linear regression was conducted to test the best predictors of BAS Arithmetic Standard Score. BAS Spelling Standard Score, VSWM

Standard Score, Age, WISC Comprehension Scaled Score, and WISC Arithmetic Scaled Score were the predictor variables, and BAS Arithmetic Standard Score was the dependent variable. The overall regression was significant, with the five predictor variables explaining 48% of the variance in BAS Arithmetic Standard Score, adjusted R^2 = .48, $F(5, 33)$ = 8.09, p < .000. Of the five predictor variables, VSWM Standard Score was a significant predictor of BAS Arithmetic Standard Score, β = .41, $t(33)$ = 2.71, p = .011; and so was WISC Arithmetic Scaled Score, β = .43, $t(33)$ = 2.83, p = .008. The remaining predictor variables were not significant at predicting BAS Arithmetic Standard Score.

Effects of Year Group and Gender on Number of Strategies Used when Controlling for Age and Derived Fact Strategy Set

To test whether Gender or Year Group had a significant effect on Number of Strategies Used, after controlling for Age and Derived Fact Strategy Set, a two-way univariate analysis of covariance (ANCOVA; = .05) was conducted with Year Group and Gender as the two independent variables, and Number of Strategies Used as the dependent variable. Age and Derived Fact Strategy Set were included as covariates.

There were no significant main effects of Gender or Year Group on Number of Strategies Used, However, Derived Fact Strategy Set did have a significant effect, $F(1, 33)$ = 11.46, p = .002, η^2 = .26.

To investigate this effect further, a new variable was computed, which grouped those in Derived Fact Strategy Set 5 in one group, and all other Derived Fact Strategy Sets in another. This was called Set Type. To test whether Set Type may account for the effect that Derived Fact Strategy Set had on Number of Strategies Used, a multivariate ANCOVA (α = .05) was conducted. Set Type was the grouping factor, and BAS Spelling Standard Score, BAS Arithmetic Standard Score, WISC Arithmetic Scaled Score, WISC Comprehension Scaled Score, BAS Arithmetic Raw Score, VSWM Standard Score, and Number of Strategies Used were the dependent variables. Age was entered as a covariate. Set Type had a significant main effect on BAS Spelling Standard Score, $F(1, 36)$ = 9.39, p = .004, η^2 = .21; BAS Arithmetic Standard Score, $F(1, 36)$ = 6.84, p = .013, η^2 = .16; WISC Arithmetic Scaled Score, $F(1, 36)$ = 7.65, p = .009, η^2 = .18; VSWM Standard Score, $F(1, 36)$ = 7.93, p = .008, η^2 = .18; and BAS Arithmetic Raw Score, $F(1, 36)$ = 8.48, p = .006, η^2 = .19. However, there was no significant effect of Set Type on Number of Strategies Used.

Predictors of the Use of Each Derived Fact Strategy Principle

To investigate the best predictors of the use of each Derived Fact Strategy principle, a series of simultaneous-entry binary logistic regressions were conducted. Since the use of the Identity and Commutativity Principles were at ceiling level, these were not investigated. The first binary logistic regression included Use of the Addend +1 Principle ('did use' vs. 'did not use') as the dependent variable, and Age, BAS Spelling Standard Score, BAS Arithmetic Standard Score, and VSWM Standard Score as the covariates. The effect of Age was significant, χ^2 = 4.05, df = 1, p = .044; and so was the effect of BAS Arithmetic Standard Score, χ^2 = 5.61, df = 1, p = .018.

The second binary logistic regression included Use of the Addend -1 Principle ('did use' vs. 'did not use') as the dependent variable, and kept the same covariates. The effect of Age was significant, χ^2 = 4.47, df = 1, p = .034. The final binary logistic regression included Use of the Inverse Principle ('did use' vs. 'did not use') as the dependent variable, and kept the same covariates. There were no significant effects.

Testing the Effects of Self-Rating and Liking for Maths

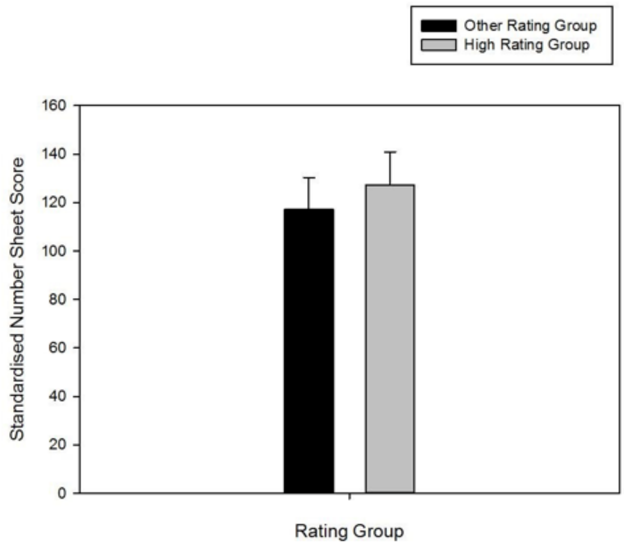
Maths rating scores were split into two groups: those who had reported the highest maths rating in one group (N = 16), and those who had reported less than the highest rating in the other (N = 23). A multivariate ANOVA (α = .05) was run, with Rating Group as the grouping factor, and BAS Spelling Standard Score, BAS Arithmetic Standard Score, BAS Arithmetic Raw Score, WISC Arithmetic Scaled Score, VSWM Standard Score, WISC Comprehension Scaled Score, Number of Strategies Used, and Age as the dependent variables. Rating Group had a significant main effect on BAS Arithmetic Standard Score, $F(1, 37)$ = 5.14, p = .029, η^2 = .12. By comparing means (see Fig. 4), we can see that BAS Arithmetic Standard Score were significantly higher for those in the High Rating Group (M = 127.13, SD = 13.82) than for those in the Other Rating Group (M = 117.22, SD = 13.16).

Maths liking scores were split into two groups: those who had reported the highest rating in one group (N = 16), and those who had reported less than the highest rating the other (N = 23). A multivariate ANOVA (α = .05) was run, with Liking Group as the independent variable, and BAS Spelling Standard Score, BAS

Arithmetic Standard Score, BAS Arithmetic Raw Score, WISC Arithmetic Scaled Score, VSWM Standard Score, WISC Comprehension Scaled Score, Number of Strategies Used, and Age as the dependent variables. Liking Group did not have a significant main effect on any of the dependent variables.

Figure 4

Comparison of significantly different mean scores in Standardised Number Sheet Score for High and Other Rating Group. Error bars show the Standard Deviation.



Discussion

Children in Year 4 obtained significantly higher scores than those in Year 2 for the raw arithmetical measures (BAS Arithmetic Score, and WISC Arithmetic Score), and there was no significant difference in their standardised scores for these measures. These results were in line with our first hypothesis, that older children would be better than younger children at all arithmetic measures, though there would be no significant differences in their standardised scores since these control for age. Furthermore, there was a significant correlation between the two standardised arithmetic measures, suggesting that these tests tap some overlapping abilities. This finding was in line with our second hypothesis. Together, these results seem to show that our measures of arithmetic ability were testing similar skills, and that these skills improve with age, thus provide a valid basis for investigating the relationships between arithmetic and other cognitive functions.

Our third hypothesis was that use of derived fact strategies would be related to other arithmetic measures, though this relationship would be stronger for written arithmetic than for mental arithmetic. We found that use of derived fact strategies, denoted by the variable Number of Strategies Used, was significantly predicted by BAS Arithmetic Standard Score, and by Derived Fact Strategy Set, though not by WISC Arithmetic Scaled Scores, thus offering partial support for this hypothesis, since use of derived fact strategies was significantly related to written, but not mental, arithmetic scores. The lack of significant relationship between mental arithmetic and derived fact strategy use is further shown in the results of the correlation analysis: use of derived fact strategies correlates with written arithmetic, but not significantly with mental arithmetic

This set of findings gives a useful insight into the relationship between children's ability to go beyond existing knowledge and perform unfamiliar calculations (as assessed by the number of derived fact strategies used) and written arithmetic ability. It could be that, although in the present study it was the mental arithmetic problems that required more reasoning, children have more experience with written arithmetic problems that include the need for numerical reasoning, while mental arithmetic tasks may more commonly emphasize fluency and fast responses An alternative explanation could be that both the Derived Fact Strategies test and the BAS Number Skills Worksheet were administered with visually presented problems, whereas the WISC Arithmetic subtest was presented verbally, with children having to abstract the mathematical problem from a 'word problem', where the sum was not always transparent. Such processes may place different demands on cognitive abilities such as executive function, thus explaining why derived fact strategy use may be related to written but not mental arithmetic. These explanations could be tested by administering a verbal analog of the Derived Fact Strategies addition test. If verbal derived fact strategy use remained related to verbal arithmetic scores, then we could perhaps conclude that the modality of the tests caused the results we observed.

Our subsequent hypotheses were related to visuo-spatial working memory. Specifically, we hypothesised that visuo-spatial working memory (VSWM) would be a strong predictor of all arithmetical measures, and would be related more to mental arithmetic (WISC Arithmetic) and use of derived fact strategies than written arithmetic (BAS Arithmetic). Furthermore, we hypothesised that VSWM would not be a strong predictor of spelling, suggesting a domain-specific input to mathematics. The results from the Pearson Correlation Coefficient analysis show that VSWM is strongly positively correlated with WISC Arithmetic Scaled Score, BAS Arithmetic Standard Score, Derived Fact Strategy Set, but not with Derived Fact Strategy Use or BAS Spelling Standard Score. Furthermore, the regression analyses revealed VSWM to be a significant predictor of BAS Arithmetic Standard Score, but not of Derived Fact Strategy Use. These findings partially support our hypotheses: VSWM is related to all arithmetic measures apart from derived fact strategy use, and is not related to spelling scores. However, it was not more strongly related to mental arithmetic than written arithmetic, since these were both still significant at the 1% level.

Our results suggest that the involvement of VSWM is specific to arithmetic, since it was not related to spelling performance. It may be that spelling is related to phonological working memory instead, since it requires manipulating phonological units of word into orthographic code. The results suggest that the domain-general measure of working memory has a specific role in arithmetic in children. Our investigation of the role of working memory in derived fact strategy use revealed no significant relationship, either in terms of Pearson Correlation Coefficient analysis, or regression analysis. This result may suggests that VSWM capacity does not influence the rate at which new derived fact strategies are acquired and used, but rather the dexterity with which they can be applied once acquired. Future research could investigate this hypothesis, by studying the effect of VSWM on the rate of acquisition of derived fact strategies, how quickly they are selected when presented with a novel arithmetic problem, and whether interrupting VSWM with a concurrent spatial memory task inhibits the use of derived fact strategies.

It may also be that derived fact strategy use requires more verbal rather than spatial working memory resources. This is brought into question by Puvandendran et al's (2016) study of an individual with Broca's aphasia and very limited verbal working memory, who nevertheless was highly successful in using derived fact strategies to compensate for retrieval difficulties. However, this patient may have been unusual in many ways, and in any case there is a difference between children who are still developing their arithmetical concepts and skills, and adults who are compensating for brain injuries that cause disruption to aspects of their existing arithmetical concepts and skills.

Our eighth and final hypothesis predicted that arithmetical self-rating would predict mental word problem solving (WISC Arithmetic) and written calculations (BAS Arithmetic), but not derived fact strategy use, since rating is more likely to be based on activities encountered in school. Our results showed that arithmetical self-rating predicted written calculation (BAS Arithmetic), and not Derived Fact Strategy Use. However, arithmetical self-rating also did not predict mental word problem solving (WISC Arithmetic). Whilst this finding is not in line with our hypothesis, since we expected self-rating to accurately reflect both written and mental aspects of arithmetic, this effect could be due to the feedback children receive whilst at school. Since most of their maths work in the classroom (as well as maths homework) is written, they are perhaps more likely to hold an accurate perception of their written arithmetic ability.

Since mental arithmetic is perhaps assessed less frequently, they may have a less clear idea about their ability in this aspect of arithmetic. Furthermore, since the perceived mathematical skill rating involved asking children explicitly about "maths", they may have interpreted this to mean specifically "written mathematics" since that is the predominant mode of mathematical teaching at school.

The children in the present study performed surprisingly well on the derived fact strategies tasks: rather better than would have been predicted from other studies (e.g. Dowker, 2014). The striking exception is that very few of them used the addition/subtraction principle, suggesting that this is particularly difficult. Most studies do concur with the present one in suggesting that the addition/ subtraction inverse relationship is acquired quite late; and is rarely used by children under the age of about 9 or 10 (Bisanz et al., 2009; Demby, 1993; Dowker, 2014; Dube, 2014). However, some studies have given more positive results. Torbeyns, Peters, DeSmedt et al (2016) found fairly frequent use of the principle by 9-to 10-year-old. Baroody et al (1983) found, unsually, that many 6-and 7-year-olds did use the addition/ subtraction inverse principle, and that indeed it appeared to precede the Addend + 1 principle. Gilmore and her colleagues (Gilmore & Bryant, 2006; Gilmore & Papadatou-Pastou, 2009; Gilmore & Spelke, 2008) found that there were considerable individual differences in primary school children's use of this principle, and that some used it successfully. It also appears that the exact nature of the problem presentation can influence performance. In Gilmore & Bryant's (2006) study, for example, included addition and subtraction in the same equations (e.g. 15 + 12 – 12), which may be easier for young children than dealing with an subtraction problem following a separate addition problem, as in the present study.

The study suggests that different aspects of arithmetic are differentially related to different contributory factors. Overall, it suggests that visuo-spatial working memory is related to both written and mental arithmetic, but not to use of derived fact strategies, although use of derived fact strategies is related to ability in written arithmetic. Furthermore, we have found that children's perceptions of their mathematics ability are associated only with their actual written arithmetic abilities, and not with their mental arithmetic abilities.

The finding that visuo-spatial working memory was significantly and independently related to both written and mental arithmetic concur with those of several studies that have suggested a significant relationship between visuo-spatial working memory and arithmetic (Ashkenazi et al., 2013; D'Amico & Guarnera, 2015; De Smedt et al., 2009; McKenzie et al., 2003; Szucs et al., 2014), and emphasize the potential importance of diagnosing and if possible intervening with visual spatial working memory difficulties at an early stage. Future studies should, however, compare the relative importance of verbal and visual-spatial working memory to the tasks discussed here. It appears from most of the studies described above that visual-spatial working memory plays a greater role in arithmetic than verbal working memory, but some studies show the opposite (Keeler & Swanson, 2001; Passolunghi & Mammarrella, 2010). The relative importance of visual-spatial and verbal working memory will depend on the aspect of arithmetic in question (Simmons et al., 2011) and is likely also to vary with age: several studies suggest that younger children's arithmetic depends more on visual-spatial working memory and older children's on verbal working memory (McKenzie et al., 2003), while Soltanou et al (2015) obtained the opposite results. Also, it may be that some studies may have failed to distinguish between the effects of visual-spatial working memory and of spatial ability as such (see Cornu et al., 2018). Clearly, a lot more research in this area is necessary.

The most significant limitation to the present study is of course the relatively small sample. Most findings were either clearly significant or non-significant, and there were few of the borderline and near-significant results that can result from underpowering, but it is important to use this study as just the springboard for studies with a much larger sample, especially as we have been studying relationships between a rather large number of variables, and wish to study still more. There is also still the potential problem that the participants could have been unusual in certain respects. Although the children in this study were from non-selective state schools, they obtained above-average standard/scaled score on most standardized tests, and performed better on the derived fact strategy tests than children in some other studies (e.g. Dowker, 2014). Also, boys performed somewhat better than girls on the mathematics tests, which is nowadays unusual at primary school level. It would be desirable to replicate the study with a larger and more varied sample.

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Appendix

Addition pre-test for Dowker et al.'s (2005) Derived Fact Strategy Addition subtest:

(1)	6 + 3	(11) 31 + 57
(2)	4 + 5	(12) 68 + 21
(3)	8 + 2	(13) 52 + 39
(4)	7 + 1	(14) 45 + 28
(5)	4 + 9	(15) 33 + 49
(6)	7 + 5	(16) 26 + 67
(7)	8 + 6	(17) 235 + 142
(8)	9 + 8	(18) 613 + 324
(9)	26 + 72	(19) 523 + 168
(10)	23 + 44	(20) 349 + 234

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Math Learning in Grade-4 and 5: What Can We Learn Form The Opportunity-Propensity Model?

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Abstract

Several factors seem important to understand the nature of mathematical learning. Byrnes and Miller combined these factors into the Opportunity-Propensity model. In this study the model was used to predict the number-processing factor and the arithmetic fluency in grade 4 (*n* = 195) and grade 5 (*n* = 213). Gender, intelligence and affect (positive affect for arithmetic fluency and negative affect for calculation accuracy) predicted math learning, and pointed to the importance of the propensity factors. We have to be careful not to interpret gender differences, since this is a social construct, our analyses pointed to the relevance of including antecedent factors in the model as well . The Implications of the study for math learning will be discussed below.

Keywords:

Mathematics, Gender, Intelligence, Propensities, Opportunities, Affect, Motivation

Introduction

Mathematics Learning

Mathematics is important in our society. Mathematics is as essential as being able to read and write (Ojose, 2011). In a longitudinal study in the United Kingdom 1700 participants were interviewed at the age of 37 about/concerning/regarding their current job satisfaction. The study revealed that people with low math skills often got low-paid jobs. About 50% of the men with low math skills had a low income, whereas this was only the case in 26% of the men with good math skills (Parsons & Bynner, 1997). Geary (2011a) confirmed the relation between poor math skills and unemployment, low chances to get promotion and low SES. Another study (*N* = 21260) revealed that children with math problems had less chance to end their secondary school with a diploma and to enter higher education (Duncan & Magnuson, 2011).

Mathematics depends on heterogeneous interrelated subskills (Fias & Henik, 2021; Kadosh & Dowker, 2015). We can distinguish calculation accuracy and arithmetic fluency. In addition, regarding mathematics, children differ as far as their motivation and affect are concerned. In a Turkish study (*N* = 789, age: 9-12year Mathematics is considered) as one of the most feared topics in education (Şahin et al., 2014).

The Opportunity-Propensity Model

Several studies explored mathematics achievement in the past, focusing on cognitive abilities as predictors (e.g., Geary et al., 2011 a&b; Landerl et al., 2021). Other studies focused on non-cognitive abilities, such as motivation (e.g., Giofrè et al., 2017) or on contextual predictors (e.g., Kaskens et al., 2020; Perera & John, 2020) of mathematics. However, by focusing on single predictors, the importance and unique explained variance of these predictors might have been overestimated.

Byrnes and Miller (2007) developed the Opportunity-Propensity (O-P) framework, aiming to differentiate between opportunity and propensity factors in an effort to explain variance and individual differences in development. They defined Propensity factors (P) as the variables that make people able (e.g., intelligence) and/or willing (e.g., motivation and affect) to learn. Opportunity factors (O) are defined as contexts and variables that expose children to learning content (e.g., home environment, classroom instruction). Antecedent (A) are defined as variables that are present early in a child's life (e.g., birth weight, birth order and gender) and explain why some people are exposed to richer opportunity contexts and have stronger propensities for learning than others (Byrnes & Miller, 2007, 2016; Wang & Byrnes, 2013).

The O-P model has been tested by the use of secondary datasets. In the first longitudinal study, researchers explained about 80% of variance through antecedent, opportunity and propensity factors in secondary school children in the United States (Byrnes & Miller, 2007). A second study with data from kindergarten up until primary school revealed additional evidence for the O-P-model with propensity factors as the strongest predictors (Byrnes & Wasik, 2009). Finally, Wang and colleagues (2013) found evidence for this model in lower-income pre-kindergarten children. A visual representation of the model can be found in Figure 1.

Antecedent Factors

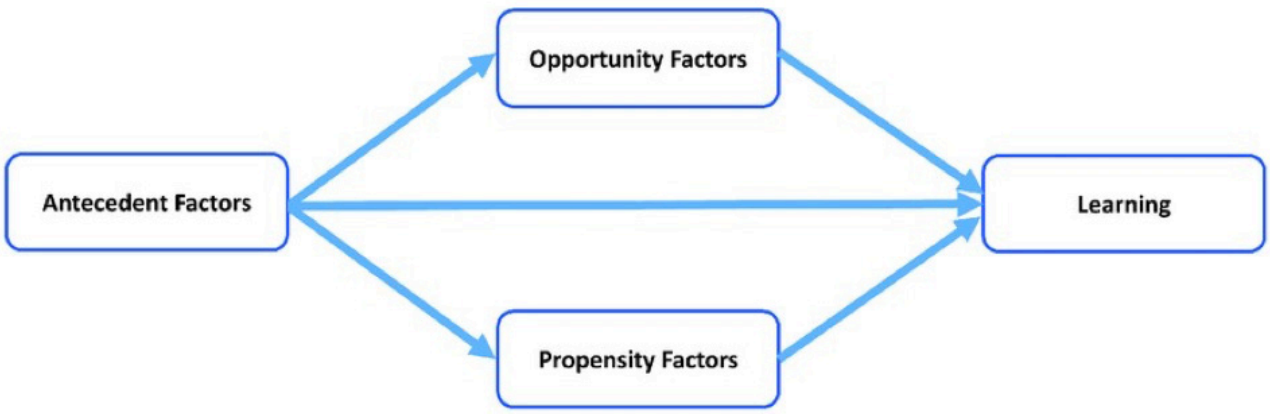
Most studies reveal that lower birth weight is related to lower levels of math performance at school-age level, with especially strong effects for extremely low birth weight (<1500 g; Chatterji et al., 2014; De Rodrigues et al., 2006; Klein et al., 1989).

Birth order seems to predict learning as well. In some studies, children who were born first, perform better in academic contexts (Belmont & Marolla, 1973; Cheng et al., 2012; Zajonc & Markus, 1975), although this was not the case in all samples (e.g., Desoete, 2008). The advantage of firstborn children has been explained by the dilution hypothesis in which the first born child takes advantage of more parental resources (at least for the time the child is only child), compared to later born children who had to share these resources (Hotz & Pantano, 2015).

Finally also gender, as a social construct might also be involved as antecedent predictor for learning. Reminding us that group differences should never be used as proof of group's superiority (Caplan & Caplan, 1997; 1999), some studies revealed that boys had better math skills than girls (Else-Quest et al., 2010; Freudenthaler et al., 2008; Lu, 2007; Lupart et al., 2004; Stoet & Geary, 2018; Zambrana et al., 2012). However other studies such as Spinath and colleagues (2010) did not find big gender differences and Byrnes and Miller (2007) and Byrnes (2020) concluded that gender could not explain much variance when other antecedent, opportunity or propensity factors that were taken into account.

Figure 1

Het Opportunity-Propensity model. Note. Adapted from "The relative importance of predictors of math and science achievement: An opportunity–propensity analysis." door J.P. Byrnes & D.C. Miller, 2007, Contemporary Educational Psychology, 32(4), p.599-629, (https://doi.org/10.1016/j.cedpsych.2006.09.002)



Opportunity Factors

There are several opportunity factors that explain variance in math learning. Teacher experience is one of this factors (Boonen et al., 2013; Byrnes, 2020; Byrnes & Miller, 2007; Clotfelter et al., 2010; Depaepe et al., 2013, Hattie, 2003). A recent study revealed that the alignment between different teachers (opportunity factor) and autonomous motivation in children (propensity factor) were the two most important predictors for the outcome variables to predict the home-learning experiences of 779 Belgian children with developmental disorders and 1443 of their typically developing peers (5-19 years) throughout the first remote learning period during the COVID-19 pandemic (Baten et al., 2022). In addition Boyd et al. (2007) and Hanushek et al. (2005) however showed that starting teachers were not always less effective compared to teachers that had more experience.

Another factor that explains variance in math learning is the exposure to the number of hours math instruction that is given (Cattaneo et al., 2016; Keith & Cool, 1992). However, in some studies the number of hours of math in class was not predictive (Aksoy & Link, 2000) or the impact differed between poor, moderate en high achieving pupils (Huebener et al., 2016).

Propensity Factors

Although some single study found no significant effect of intelligence (e.g. Jones & Byrnes), most studies demonstrated a significant relationship between intelligence (Floyd et al., 2003; Kucian & von Aster, 2015; Primi et al., 2010; Roth et al., 2015; Taub et al., 2008) and academic performance. Finally, some researchers focused on non-cognitive predictors (Schoenfeld, 1983) such as motivation (Deci & Ryan, 2008a&b; Froiland & Worrell, 2016; Ryan & Deci, 2000) and well-being or positive and negative affect (Awang-Hashim et al., 2015; Diener, 1984; Diener et al., 2005; McLeod, 1990; McLeod & Adams, 1989; Peixoto et al., 2016; Pekrun et al., 2006). In a meta-analysis, Taylor and colleagues (2014) highlighted a positive relationship between autonomous motivation (where the force to fulfill a task is internal, e.g., passion) and general school achievement, in addition to a negative relationship between controlled motivation (where the force to fulfill a task is external, e.g., reward-related) and academic achievement. This relationship was confirmed by several studies (Nurmi & Aunola, 2005; Pantziara & Philippou, 2014; Schneider & Bös, 1985; Steinmayr & Spinath, 2009). In addition also well-being can be considered a propensity factor, since it makes people willing and able to learn. Positive and bidirectional relations between subjective well-being and academic performance were found. Students with higher levels of subjective well-being (and more positive emotions than negative emotions) had better academic performance and vice versa. Furthermore, higher perceptions of own academic competence were predictive of better academic achievement and vice versa (Arefi et al., 2014) which confirmed the reciprocal-effects model between academic self-concept and academic achievement (Seaton et al., 2015).

Current Study

Although there is plenty of evidence for this model (Byrnes & Miller, 2016, 2007; Byrnes & Wasik, 2009; Wang & Byrnes, 2013) from secondary datasets, the model remains unknown and there is little research from primary data simultaneously tapping the antecedents, opportunities and propensities empirically in children explaining their mathematical achievement. Recently a PhD study was set up at Ghent University to explore how mathematics learning is related to factors described in the opportunity-propensity model using primary datasets. This resulted in a cross-sectional study combining antecedent, opportunity, and propensity factors in 114 numbchildren (Baten & Desoete, 2018) and in 30 adults (Baten & Desoete, 2021) as well as in an intervention study (Baten et al., 2020). The current study is an attempt to replicate the usefulness of the model on a larger sample of children (n = 408). It might seem unimportant to include antecedent factors as predictors, since these are clearly factors over which educators have no control. However including antecedent factors is essential not to overestimate the predictive value of opportunity and propensity factors in the model. As such, this study contributes to theory-building about mathematical learning. The study has two research questions (RQ).

Rq1: What Factors are Related to Proficient Mathematics in Grade 4 and 5 in Flanders?

The study investigated antecedents factors related to mathematics in grade 4 and 5. We studied the influence of gender (Freudenthaler et al., 2008; Lu, 2007; Lupart et al., 2004; Zambrana et al., 2012), birth order (Belmont & Marolla, 1973; Cheng et al., 2012; Hotz et al., 2015; Zajonc & Markus, 1975) and birth weight (Breslau et al., 2004; Chatterji et al., 2014; De Rodrigues et al., 2006; Klein et al., 1989).

In addition the study included opportunity factors related to mathematics in grade 4 and 5. We studied if the number of years of experience in teaching (Boonen et al., 2013; Clotfelter et al., 2010) and the instruction time (Aksoy & Link, 2000; Keith & Cool, 1992) predicted math proficiency in Flanders.

The study also included propensity factors, such as intelligence (Floyd et al., 2003; Kucian & von Aster, 2015; Primi et al., 2010; Roth et al., 2015; Taub et al., 2008), positive and negative affect related to mathematics (Peixoto et al., 2016; Pekrun, 2006) and motivation

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(Nurmi & Aunola, 2005; Pantziara & Philippou, 2014; Steinmayr & Spinath, 2009).

Finally, in line with previous studies (Baten & Desoete, 2018; Fias & Henik, 2021) the impact of these factors on calculation accuracy and on fact retrieval fluency was studied.

Rq2: Are there Gender Differences as far as Mathematics and the Antecedent, Opportunity and Propensity Factors are Concerned?

We studied in line Else-Quest et al. (2010), Lu (2007), Zambrana et al. (2012) if there were gender differences in this sample, and expected in line with Bakhiet et al. (2015) no gender differences on intelligence, but higher intrinsic motivation (Skaalvik & Skaalvik, 2004) and more positive affect (Rubinsten et al., 2012) related to mathematics in boys.

Method

Participants

408 children in total participated in this cross-sectional study. The sample included 195 children (79 boys, 116 girls) from grade 4 and 213 children (84 boys, 129 girls) from grade 5. The age of the children differed from 9 till 12 years. The sample included 15 children with dyscalculia (3.68%), 27 children with dyslexia (6.62%) and 17 multilingual children (4.17%).

Procedure

After parents agreed to the participation of their children, an appointment for the actual research was made. Sessions lasted about 90 minutes while tests and questionnaires were administered individually to each child. Testing happened in a location chosen by the parents. The researcher gave standardized instructions and was available to answer questions.

Instruments

Antecedent and opportunity factors were measured through questionnaires. More specifically, for the O factors, teachers were asked how many years of experience they had in teaching mathematics and how many hours of mathematical instructions the children received per week (teaching hours).

To measure A factors, parents were asked about their aspirations regarding the mathematical abilities of their children. They had to reflect on the score they wanted their child to have at the end of the current school year (in percentage). Additionally, information on birth order and birth weight of the child was collected.

With regards to the P factors the following instruments were used.

Intelligence was measured using an abridged Dutch version of the Wechsler-Intelligence-Scale for Children-III (WISC-III-NL; Kort et al., 2005). The total intelligence quotient or IQ (M = 100; SD = 15) was obtained by combining the separate scores on the following subtests: Vocabulary, Similarities, Picture Concepts, and Block Design. The reliability of this short form was .92 and the distribution of total IQ-scores calculated with the short form did not significantly differ from the distribution of the scores on the full intelligence test (Grégoire, 2000). Cronbach's α of the total IQ in the current sample was .795.

Motivation for mathematics was measured with the Dutch version of the Academic Self-Regulation Scale (Vansteenkiste et al., 2009) which consists of 24 questions which allow the calculation of the level of autonomous and controlled academic motivation. As suggested by the authors, the introduction for the questions was changed from 'I am motivated to study because...', to 'I am motivated to study mathematics because ...' in order to measure motivation with regards to mathematics specifically. The child had to respond on a 5-point Likert scale to statements such as 'because I find this an important goal in my life' as an index of autonomous motivation and 'because other people (e.g. parents, friends, teachers) oblige me to do so' to measure controlled motivation. The score for each scale was calculated by averaging the score on the items belonging to that scale. Cronbach's α for this sample was .86 for autonomous and .72 for controlled motivation.

Subjective well-being was determined through the Dutch version of the Positive and Negative Affect Schedule (PANAS; Watson et al., 1988; translated by Engelen et al., 2006). Children indicated on a 5-point Likert scale how many negative (e.g. guilt and sadness) and positive (e.g. success and interest) emotions they experienced on a regular school day. Scores were calculated for the level of positive affect and the level of negative affect by averaging the score on 10 items. Cronbach's α for this sample was .85 for positive affect and .77 for negative affect.

Arithmetic fluency (fact retrieval speed) and calculation accuracy investigated as outcome measures.

To measure the arithmetical fluency, the Arithmetic Number Fact Test (de Vos, 2002) was used. Children had to solve as many additions (e.g. '7+2'), subtractions (e.g. '6-5'), multiplications (e.g. '5x8'), divisions (e.g. '27:9') or a mix of these exercises as possible within five minutes. The number of correct answers was used as outcome measure. This test has been standardized for Flanders on a sample of 10059 children. The psychometric value of the test has been demonstrated with a Cronbach's alpha of .900 (Desoete and Roeyers, 2005). For this sample Cronbachs α was .92.

To measure the calculation accuracy of the child, the Kortrijkse Rekentest Revisie (KRT-R; Baudonck et al., 2006) was administered. This test evaluates the conceptual understanding and the proficiency or accuracy needed to solve 90 exercises in a number-problem or word-problem format (e.g., '283 times more than -71 is ...'; '27681:90 = ...'; 'Wim has 4.8kg of flour. Jan has a double amount of flour. How many flour do Jan and Wim have together?') without a time limit. The number of correct answers was calculated as outcome measure. The internal consistency for this sample was Cronbach's α = .84.

Statistical Analyses

Statistical Package for the Social Sciences (SPSS), version 27 was used to analyse the data. First Spearman correlations were calculated.

To answer the first research question, multivariate hierarchic regression analyses were conducted. The multivariate hierarchic version was used. For fluency's sake rough data were used. For calculation accuracy z-scores were calculated since the test for grades 4 and 5 had other items. All analyses were conducted if/whenever the conditions to conduct parametric tests were fulfilled (Field, 2009).

For the second research question the condition of multivariate normality was not fulfilled, so independent samples t-tests were used. Bias-corrected and accelerated (BCa) lower and upper confidence intervals were computed using bootstrapping as computer-intensive resampling techniques that involved 1000 bootstrap samples based on the original observations in this study, as robust hypothesis testing of differences. The 1000 bootstrapped means were put in order, from lowest to highest, and the central 95% of values were used to form the confidence interval, using SPSS 27.

Results

Descriptive Statistics

Table 1 gives an overview of all correlations and the descriptive statistics that are given. Both math components (arithmetic fluency and calculation accuracy) correlated significantly (r = .43, p < .01). In addition fluency correlated with intelligence (r = .20, p < .01), positive affect (r = .32, p < .01) negative affect (r = -.19, p < .01) and autonomous motivation (r = .30, p < .01). For calculation accuracy similar results were found. Intelligence (r = .49, p < .01), positive affect (r = .33, p < .01), negative affect (r = -.31, p < .01) and autonomous motivation (r = .33, p < .01) correlated significantly with calculation accuracy. In addition birth weight correlated significantly with positive affect (r = -.14, p < .01) and there was a significant correlation between birth order and birth weight (r = .18, p < .05). The propensity factors also correlated significantly with each other.

Research question 1: What antecedent-, opportunity- and propensity factors are related on math in grade 4 and 5?

To answer this multiple question, in line with Field (2009, p 212) hierarchic regressions were conducted.

Table 1
Correlations between the variables and descriptive statistics

		M(SD)	1	2	3	4	5	6	7	8	9	10	11
1	Fluency	105.33 (20.96)											
2	Calculation	.00 (.99)	.43**										
3	Gender	-	-.16**	-.25**									
4	Birth weight	3342.36 (515.53)	.03	.07	-.15**								
5	Birth order	1.89 (1.07)	.05	-.00	-.06	.18*							
6	Experience T	17.28 (10.87)	.05	.04	-.01	.07	.05						
7	Hour math	6.15 (.90)	-.01	.04	.02	.04	-.03	.12					
8	Intelligence	0.00 (1.00)	.20**	.49**	-.05	.08	.00	.07	-.02				
9	PA	3.31 (0.72)	.32**	.33**	-.14**	-.02	-.06	.04	.00	.13**			
10	NA	1.71 (0.54)	-.19**	-.31**	-.02	-.03	-.01	-.03	.06	-.27**	-.42**		
11	Aut. mot.	3.48 (.88)	.30**	.33**	-.11*	-.03	-.03	.04	-.04	.17**	.74**	-.42**	
12	Cont. mot.	2.61 (.78)	.01	-.09	-.05	.30	.06	-.06	.02	-.17**	-.13**	.32**	-.17**

*Note: ** p < .01, * p < .05, Gender coded as 0 = male and 1 = female, Birth weight = birth weight in gram, , Experience T = experience teacher measured in number of years teaching, Hours math = number of hours per week a teacher teaches math, Intelligence = z-score on the Raven, PA = positive affect, NA = negative affect, Aut. mot. = total autonomous motivation, Cont. mot. = total controlled motivation*

In step 1 all antecedent factors were added. In step 2 all opportunity factors were added. Finally the opportunity factors were added..

Arithmetic Fluency

The antecedent factors in step 1 predicted a significant percentage of variance in arithmetic fluency (F(3, 310) = 3.37, p = .019). Especially gender was important. Boys were better in arithmetic fluency compared to girls. Adding the opportunity factors to the model (in step 2) did not improve the model (F(5, 308) = 2.07, p = .068). The teacher experience and the number of hours of instruction were no significant predictors for arithmetic fluency. Adding propensity factors (in step 3) improved the model, with 13% more explained variance (F(10, 303) = 5.97, p < .001). There was an explained variance of 14% with intelligence and positive affect that predicted arithmetic fluency. For more information, see Table 2.

Calculation Accuracy

In step 1 the antecedent factors explained 5% of the variance (F(3, 311) = 6.09, p < .001) in calculation accuracy. Gender was no significant predictor.

Adding opportunity factors (in step 2) made the model significant (F(5, 309) = 3.88, p = .002). However the experience of the teacher and the number of hours mathematics instruction were no significant predictors of calculation accuracy skills of children.

Adding propensity factors improved the model with 33% explained variance (F(10; 304) = 19.36, p < .001). Intelligence and negative affect were significant predictors of calculation accuracy. For more information, see Table 3

To conclude, the included opportunity factors were no significant predictors, whereas propensity variables explained 13% of the variance of arithmetic fluency and 33% of the variance of calculation accuracy. Intelligence was a significant predictor for fluency and accuracy, whereas positive affect only influenced arithmetic fluency.

Research question 2: Are there gender differences on mathematics and on the antecedent, opportunity and propensity factors?

Independent sample t-tests were used to look for gender differences, see Table 4.

Boys were better in mathematics compared to girls. They also experienced more positive affect. There were no significant gender related differences on opportunity factors. Boys in this sample had a higher birth weight compared to girls.

Table 2

Results of hierarchic multiple regressions on the antecedent, opportunity- en propensity factors of arithmetic fluency

	Arithmetic fluency							
Variable	R^2	*Adj.* R^2	ΔR^2	*B*	*SE B*	β	*t*	*p*
Step 1	.03	.02	.03					
Gender				-6.96	2.37	-.17	-2.94	.003**
Birth weight				.00	.00	.03	.53	.599
Birth order				.37	1.03	.02	.36	.719
Step 2	.03	.02	.00					
Gender				-7.03	2.37	-.17	-2.96	.003**
Birth weight				.00	.00	.03	.54	.591
Birth order				.31	1.04	.02	.29	.768
Teacher experience				.03	.11	.01	.26	.797
Hours math instruction				-.64	1.23	-.03	-.52	.603
Stap 3	.16	.14	.13					
Gender				-4.81	2.28	-.11	-2.11	.036*
Birth weight				.00	.00	.01	.28	.780
Birth order				.34	.98	.02	.34	.732
Teacher experience				-.01	.10	-.01	-.11	.909
Hours math instruction				-1.16	1.16	-.05	-1.00	.320
Intelligence				3.08	1.11	.15	2.78	.006**
Positive affect				6.58	2.34	.23	2.81	.005**
Negative affect				1.23	2.39	-.03	-.51	.607
Autonomous motivation				2.26	1.94	.09	1.17	.243
Controlled motivation				1.77	1.48	.07	1.19	.235

*Note. * p < .05, ** p < .01*

Table 3

Results of the hierarchic multiple regression analyses of the antecedent, opportunity- en propensity factors on calculation accuracy

Calculation accuracy								
Variable	R^2	*Adj.* R^2	ΔR^2	*B*	*SE B*	β	*t*	*p*
Step 1	.05	.05	.05					
Gender				-.45	.11	-.22	-4.02	<.001**
Birth weight				.00	.00	.05	.83	.409
Birth order				-.01	.05	-.01	-.14	.888
Step 2	.06	.04	.00					
Gender				-.45	.11	-.23	-4.03	<.001**
Birth weight				.00	.00	.04	.74	.947
Birth order				-.01	.05	-.01	-.21	.397
Teacher experience				.01	.00	.06	1.09	.598
Hours math instruction				.00	.06	.00	-.01	.762
Step 3	.39	.37	.33					
Gender				-.39	.09	-.19	-4.16	<.001**
Birth weight				.00	.00	-.00	-.07	.947
Birth order				-.03	.04	-.04	-.85	.397
Teacher experience				.00	.00	.02	.53	.598
Hours math instruction				-.01	.05	-.01	-.30	.762
Intelligence				.43	.04	.45	9.53	<.001**
Positive affect				.15	.10	.11	1.54	.125
Negative affect				-.30	.10	-.16	-3.05	.002**
Automous motivation				.12	.08	.10	1.48	.141
Controlled motivation				.10	.06	.08	1.61	.109

Note. * $p < .05$, ** $p < .01$

Discussion

Mathematics is important in our society (Duncan & Magnuson, 2011; Geary, 2011a & b; Ojose, 2011). The Opportunity-Propensity (O-P) model (Byrnes, 2020; Byrnes & Miller, 2007; Wang et al., 2013) integrates predictors of learning, and helps gaining insight into how predictors are interrelated, and whether some are more important than others.

Answering the first research question and looking at the antecedent factors, in line with some previous studies (Baten & Desoete, 2018; Desoete, 2008), but in contrast with other studies on birth weight (Breslau et al., 2004; Chatterji et al., 2014; De Rodrigues et al., 2006; Klein et al., 1989) and birth order (Belmont & Marolla, 1973; Cheng et al., 2012; Hotz et al., 2015; Zajonc & Markus, 1975) these antecendent factors could not significant explain variance in fact retrieval or calculation accuracy in our sample. However gender as antecendent factor, attributed to the variance in both math components. In line with previous studies (Freudenthaler et al., 2008; Lu, 2007; Lupart et al., 2004; Zambrana et al., 2012) boys were more proficient in mathematical fluency and in calculation accuracy in grade 4 and 5.

Looking at opportunity factors, the present study could not confirm significant predictors for math proficiency. The experience of the teacher nor the number of hours of instructions were significant predictors of variability in mathematics. The fact that experience was no significant predictor is in contrast with previous studies (Baten & Desoete, 2018; Boonen et al., 2013; Clotfelter et al., 2010), but the fact that the number of hours of instruction was not significant confirmed previous findings in Flanders (Baten & Desoete, 2018). It might be that not only the quantity of instruction, but especially the quality of instruction matters. Moreover, to engage in mathematics may also have more to do with what is happening outside the classroom than in for many students. Additional studies are needed including measures such as school attendance, parental educational level etc.

Looking at propensity factors, in line with previous studies in Flanders (Baten & Desoete, 2018), motivation did not predict math proficiency in grade 4 and 5. These findings are in contrast with the findings of Steinmayr and Spinath (2009) who found that higher motivation resulted in better math results. Intelligence was a significant predictor for math fluency and calculation accuracy, confirming previous studies (Baten & Desoete, 2018; Floyd et al., 2003; Kucian & von Aster, 2015; Primi et al., 2010; Roth et al., 2015; Taub et al., 2008). In this study there was a significant effect of positive affect on math fluency and a significant effect of negative affect on calculation accuracy, where in a previous study we found the reversed picture (Baten & Desoete, 2018).

Table 4

	Boys	Girls
Arithmetic Fluency (t_{313} = 3.32, *p* = .001). M [95% CI] *SD*	109.63_a [106.21; 113.03] (22.30)	102.49_b [100.15; 104.81] (19.56)
Calculation accuracy (t_{406} = 5.08, *p* < .001) M [95% CI] *SD*	0.30_a [0.14; 0.44] (0.97)	-0.20_b [-0.32; -0.08] (0.97)
Birth weight (t_{352} = 3.08, *p* = .002), M [95% CI] *SD*	3444.95_a [3367.42; 3523.64] (489.38)	3274.44_b [3207.80; 3341.64] (522.25)
Birth order (t_{403} = 1.65, *p* = .100). *M* [95% CI] *SD*	1.99 [1.81; 2.17] (1.17)	1.82 [1.70; 1.96] (0.98)
Teacher experience (t_{382} = 0.15, *p* = .878) *M* [95% CI] *SD*	17.38 [15.67; 19.06] (11.02)	17.21 [15.88; 18.65] (10.78)
Hours math instruction (t_{376} = 0.32, *p* = .751) *M* [95% CI] *SD*	6.17 [6.02; 6.35] (1.14)	6.14 [6.04; 6.24] (0.70)
Intelligence (t_{405} =0.86, *p* = .393) *M* [95% CI] *SD*	0.05 [-.10; .20] (1.03)	-0.03 [-.16; .08] (0.98)
Positive affect (t_{405} = 2.74, *p* = .006) *M* [95% CI] *SD*	3.43_a [3.31; 3.55] (0.74)	3.23_b [3.14; 3.32] (0.70)
Negative affect (t_{405} = 0.63, *p* = .529) *M* [95% CI] *SD*	1.74 [1.66; 1.83] (0.56)	1.70 [1.64; 1.77] (0.52)
Autonomous Motivation (t_{398} = 1.88, *p* = .061) *M* [95% CI] *SD*	3.58 [3.44; 3.73] (0.93)	3.41 [3.32; 3.52] (0.85)
Controlled Motivation (t_{397} = 0.92, *p* = .357) *M* [95% CI] *SD*	2.66 [2.54; 2.78] (0.78)	2.58 [2.48; 2.68] (0.78)

Note. 95% CI = 95% Confidence Interval

When comparing antecedent, opportunity and propensity factors, propensity factors were the strongest predictors for both math components. This finding confirmed a previous study on elementary school children (Baten & Desoete, 2018).

Answering the second research question, in line with earlier studies (Else-Quest et al., 2010; Stoet & Geary, 2018) boys were more proficient on mathematics compared to girls. However, these results have to be interpreted carefully since gender is increasingly being thought of as a social construct, rather than a biological one and some researchers point to the fact that analyzing sex differences might even be potentially harmful. These finding should therefore not be seen as proof of a more powerful group's superiority, but only as one of the antecedent factors in the O-P model. In addition, boys in this study had, in line with the findings of Voldner et all (2009) a higher birth weight. In contrast with Simonton (2008), we did not find evidence for differences in birth order. As expected there were no significant gender related opportunity differences. Looking at gender related propensity predictors, boys and girls only differed on positive affect, with boys having more positive feelings about mathematics compared to girls. This finding is in contrast with earlier studies (Ghasemi & Burley, 2019) were no gender differences were found. Boys and girls did not differ in this study on intelligence or on motivation.

This study has some limitations. First, there was no gender balance in the sample. More girls participated to the study. The second limitation was the cross-sectional design of the study and the fact that not all relevant factors of the opportunity-propensity model could be included. Finally Figure 1 might be an simplified version of the O-P model, since there is also a relationship from opportunities to propensities (Wang et al., 2013). Thus opportunity and propensity factors might not be as separate as Figure 1 would presume. Additional studies should include all relationships. In addition we should conduct longitudinal studies including also other O-P predictors such as teacher quality, school attendance, language fluency, SES, parental education level etc..

However the present analyses confirmed the value of the O-P model and gave us information of a rather large sample of children (*N* = 408) and their teachers.

Conclusion

In summary, our findings suggest two general conclusions. First, gender as antecedent factor in the Opportunity Propensity model (Byrnes & Miller, 2007) remains important. Gender friendly targeted instruction and giving all students the opportunity to engage in mathematics may be a educationally important. Second, especially propensity factors, such as intelligence and positive and negative affect explain variance in mathematical proficiency.

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The Heterogeneity of Mathematical Learning Disabilities: Consequences for Research and Practice

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Abstract

This paper argues why children with Mathematical Learning Disabilities (MLD) do not form a unitary group. Instead, they should be regarded as individuals with unique profiles of strengths and weaknesses that explain their mathematical difficulties. To build this argument, we shortly recapitulate the research on MLD, which has mainly been focused on characterizing the group of children with MLD– as compared to control groups. However, these general characteristics are not applicable to all children with MLD. Furthermore, attempts to define separate, relevant subgroups merely failed. Based on some recent studies, we show how individual profiles of strengths and weaknesses might help in understanding the specific mathematical difficulties of a child. We propose a new multidimensional framework of MLD, in which both strengths and weaknesses are recognized. We argue that both research and practice are in need of further research that takes individual differences into account.

Keywords:

Mathematical Learning Disabilities, Dyscalculia, Strengths-and-weaknesses, Multidimensional Approach

Introduction

Mathematical Learning Disabilities (MLD) refer to specific, severe and persistent difficulties that children can encounter in learning mathematics. In general, children with MLD have both difficulties with learning and remembering arithmetic facts and difficulties in executing calculation procedures (Landerl et al., 2004). Whereas MLD is a common term, different terminologies have also been used for similar concepts, such as mathematical learning difficulties (e.g., De Smedt, & Gilmore, 2011; Mazzocco, 2007), mathematical learning disabilities (e.g., De Smedt et al., 2012; Desoete, 2007; Geary, 2011; Mazzocco, 2007; Szűcs, 2016), mathematical learning disorders (e.g., Desoete & De Weerdt, 2013) and dyscalculia (e.g., Butterworth et al., 2011; Van Luit, 2019) or developmental dyscalculia (e.g., Butterworth, 2008; Dehaene et al., 1993; Shalev, 2004; Van Luit, & Toll, 2018). It should be noted that these terminologies might seem interchangeable, and are sometimes used as such. However, in the literature, some (variable) distinctions are often made to differentiate between the various terms. For example, different nuances to the term MLD usually indicate gradations in the severity of the math learning problems (e.g., a difficulty is less severe than a disability), whereas developmental dyscalculia might indicate more specific impairments such as a core deficit in number sense (Butterworth et al., 2011; Dehaene et al., 1993; Shalev, 2004).

Next to differences in terminology, there are different definitions and operationalisations of MLD in the literature. A closer look on empirical studies shows that most often MLD samples are only based on the seriousness of math difficulties (e.g., performance below a certain cut-off score), although sometimes accompanied by the criterion of persistence (e.g., the problems exist at least for a certain period; Kroesbergen et al., 2021). Comorbid learning, behavioral or developmental problems are excluded in some studies (i.e., specificity criterion), but are the subject of study in others. The fact that different cut-off criteria are used (ranging from percentile 2 to 40; Kroesbergen et al., 2021), makes the comparison of different studies even more difficult, and the implications for practice at least vague. Nevertheless, despite these limitations, some conclusions can be drawn from former research about MLD.

A growing body of research has tried to find explanations for the phenomenon of MLD, based on the assumption that underlying cognitive deficits cause the specific mathematical problems. In the next section, we will give an overview of this line of research, mainly based on review studies and meta-analyses. The existing literature on underlying deficits has mainly focused on characterizing children (or adults) with MLD, by comparing groups of children with and without MLD. Other studies have used more descriptive methods. An interesting idea in this line of research is that there is not one type of MLD, but that several subtypes exist (e.g., Geary, 2004; Moeller et al., 2012), which might have different origins or manifestations. Furthermore, different lines of MLD-related research have turned their focus towards the brain, in order to investigate the neural underpinnings of MLD, using neuroscientific methods. Despite the promising advances in this field, no clear brain structures or networks have yet been identified in children with MLD. For this reason, we will not include the neural models in our current paper.

After reviewing these different lines of research, we propose an alternative view on MLD, which takes not only into account the possible deficits, but also the (compensating) strengths that could potentially be related to children's math performance. Importantly, this approach assumes that there is no inherent distinction between children with and without MLD. Rather, we propose that children's mathematical performance should be regarded as a continuous scale ranging from very poor to very strong math performance. Hence, a clear distinction between children with and without MLD is at least difficult if not impossible to make. Following this alternative model, the present paper concludes with implications for both research and practice.

Characterizing MLD: Results from Group Comparison Studies

A large body of research has shown that mathematical performance is related to a number of domain-general and domain-specific cognitive skills. The most salient cognitive skills involved in math learning are number sense, working memory, attention, processing speed, and phonological processing (e.g., Geary, 2004; Mammarella et al., 2021; Peng et al., 2018). Not surprisingly, these are also the skills that are suggested to play a role in MLD. Many studies on the characteristics of MLD have compared MLD groups with control groups, to find in what way children with MLD differ from children without MLD. Based on these studies, some conclusions can be drawn about the cognitive characteristics of MLD in general, in which a distinction can be made between domain-specific and domain-general cognitive skills.

Number sense - the only domain-specific skill related to mathematics and MLD - can be defined as the ability to recognize and understand non-symbolic numerosity (quantities) and symbolic numbers (number words and Arabic digits), and mapping between these numerical representations (Dehaene et al., 2003; Geary, 2011). More comprehensive definitions of number sense also include skills like counting, nonverbal calculations and number patterns (e.g., Jordan et al., 2007). Berch (2005) proposed to make a distinction between lower order and higher order number sense. Lower order number sense refers to the intuitive perceptions of quantity, as described above. Berch considers higher order number sense much more complex and multifaceted, comprising a deep understanding of mathematical principles and relationships and a high degree of fluency and flexibility with operations and procedures. Number sense is thought to play a crucial role in MLD (e.g., Jordan et al., 2007). More specifically, a core deficit in numerosity processing has been proposed to underlie MLD, or at least a specific form of MLD, namely pure or developmental dyscalculia (Butterworth, 1999; Mazzocco, et al., 2011). It should be noted however, that the core-deficit view has been criticized by scholars, because it might not be a deficit in processing numerosity itself, but in accessing numerical meaning from symbolic digits (De Smedt & Gilmore, 2011; Rousselle & Noel, 2007). In a meta-analysis on the differences between MLD and typically developing (TD) children in number sense, effect sizes for symbolic skills were indeed significantly larger than for non-symbolic skills (Schwenk et al., 2017). These results were replicated in a recent meta-analysis (Kroesbergen et al., 2021). The latter study, however, also found that effect sizes for higher-order number sense skills were even higher. It is interesting to note, that according to the meta-analyses by Kroesbergen et al. (2021) and Schneider et al. (2017), the differences in number sense between people with and without MLD, seem to decline with age.

Studies that compared children with MLD to typically developing control groups, have pointed to several domain-general cognitive skills that are weaker in children with MLD. An extensive meta-analysis on 75 cognitive profiling studies on MLD showed that these differences are especially apparent in working memory and processing speed, but also in phonological skills and attention (Peng et al., 2018). Other domain general skills that have been found to be weaker in groups of children with MLD are spatial skills (e.g. Peters et al., 2020; Träff et al., 2020), visual perception (e.g. Cheng et al., 2018), ordering/ order processing (e.g. Morsanyi et al., 2018; Sasanguie et al., 2017), inhibition (e.g. Szűcs et al., 2013), number-specific executive function (Wilkey et al., 2020), and logical/ non-verbal reasoning (Huijsmans et al., 2022; Träff et al., 2020). The main weaknesses found are shortly elaborated below.

Working memory involves the temporal storage, processing, and recollection (i.e., the executive function of updating) of verbal and visuospatial information (Alloway et al., 2009; Passolunghi, & Siegel, 2004). A vast amount of research has identified working memory as a domain general cognitive factor in learning mathematics. Strong working memory skills facilitate stepwise solving multiple-component math problems. Poor working memory skills on the other hand, have been associated with MLD (e.g. David, 2012; Klesczeweski et al., 2018). An interesting finding is that especially the processing of numerical information in working memory (as compared to non-numerical verbal information) is often impaired in children with MLD (e.g. David, 2012; Peng & Fuchs, 2016; Peng et al., 2012; Raghubar et al. 2010; Wilkey et al., 2020), which also points to a domain-specific deficit. However, visual-spatial working memory seems to be more affected than verbal working memory in children with MLD (David, 2012), and especially spatial working memory (e.g., Mammarella et al., 2018, Szűcs et al., 2013).

Attention refers to the allocation of cognitive resources to relevant stimuli (Posner & Petersen, 1990) and has often been found to be impaired in children with MLD (e.g. Peng et al., 2018). Attention might especially be necessary to support the executive process when doing (complex) calculations, especially when arithmetic facts are not (yet) automatized (Peng et al., 2018). However, attention is also required in learning and automatizing arithmetic facts, which requires an active engagement.

Processing speed can be defined as the speed at which a person is able to encode, transform, and retrieve information (Conway et al., 2002). Processing speed is found to play a role in the development of mathematics as it facilitates the temporary storage on answers of simple sums and counting words in working memory (Geary, 1993). In their meta-analysis, Peng et al. (2018) showed that shortcomings in processing speed and short-term memory were related to problems in higher-level cognitive skills such as working memory and attention, supporting the idea that processing speed plays a central role in explaining mathematical deficits in all children with MLD.

Phonological processing has emerged as a domain general factor in mathematics as well. Phonological awareness (Vellutino et al., 2004) and rapid naming (Donker et al., 2016; Willburger et al., 2008) have been identified as relevant components of phonological processing. Quick access to verbal codes stored in long-term memory (i.e., rapid automatized naming) that correspond to number facts, and effective recognition and manipulation of those verbal codes (i.e., phonological awareness) are required for arithmetic fact retrieval (Simmons, & Singleton, 2008). Although weaker phonological skills have been found in children with MLD (e.g., Peng et al., 2018), deficits in phonological skills are most often found in children with both mathematical and reading difficulties and could thus possibly explain the frequent comorbidity between mathematics and reading problems (Peng et al., 2018; Slot et al., 2016). It has been suggested that affected children may have difficulties with fact retrieval, which interferes with their mathematical abilities (Landerl et al., 2009).

Although in general some characteristics of MLD can be determined, the heterogeneity within the group of MLD is large. To demonstrate this heterogeneity, we will describe some recent case studies, which have demonstrated that very different (specific) factors may play a role in different people with mathematical learning problems. This supports the idea that not all people with MLD experience the problems described above.

Characterizing MLD: Results from Case Studies

Several case studies on MLD have been described in the literature. They all have a specific focus, and inherently different descriptions of MLD. All of them measured at least some form of number sense as potential underlying deficit. In addition, domain general skills are described as possible explaining variables. First, we will shortly summarize these case studies.

De Visscher and Noël (2013) describe a case study of an adult woman with a specific form of dyscalculia, in combination with high general cognitive capacities. She has specific arithmetic fact retrieval deficit, as shown in very long reaction times (but accurate performance), most visible in multiplication facts. The authors used this case to investigate the specific hypothesis of hypersensitivity-to-interference in memory. Being highly sensitive to interference means experiencing difficulties in retrieving the exact context of similar items which have been processed recently. In the case of arithmetic facts, the context is the problem, which has to be associated with the answer, all consisting of numbers. The results of this woman were compared to a reference group of 11 women matched on educational level and age. Remarkably, this woman performed above average on other cognitive skills related to MLD: attention, executive functions, phonological processing, and verbal and visual working memory. Number sense, as measured with a dot estimation task and a comparison task, was also not impaired. However, she did show a high sensitivity to interference, as measured with learning-associations tasks. Thus, in this specific case, high sensitivity to interference caused the specific mathematical difficulty.

Two other case studies focused on specific forms of number sense, namely subitizing and number lines (Moeller et al., 2009; Van Viersen et al., 2013). Both studies investigated in depth the performance of children with MLD on the respective tasks, applying eye-tracking. Moeller and colleagues (2009) report on two 10-year-old boys with dyscalculia, without problems in reading, general cognitive abilities, or attention. They investigated the subitizing skills (range 1-8) of these boys, and compared these to a reference group of 8 age-matched typically developing children. The boys were impaired in subitizing (range 1-3) as well as enumeration in the counting range (4-8). By applying eye-tracking, the researchers were able to show that even with the smallest numbers, the boys often used counting strategies instead of subitizing. They conclude that the problem lays in quick automatic and parallel encoding of non-symbolic quantities.

Van Viersen and colleagues (2013) applied eyetracking to investigate the strategies of a 9-year-old girl with MLD on a symbolic and a nonsymbolic numberline task (compared to a reference group of 10 typically developing children). In addition to poorer performance on the numberline tasks (0-100 and 0-1000), the child also performed lower on visual-spatial working memory, but showed average performance on verbal working memory. The analyses of eyetracking showed that she used less clear strategies and that her strategies were often less efficient and atypical as compared to those of the reference group.

Other case studies have focused on both domain-specific and domain-general cognitive factors related to MLD (Davidse et al., 2014; Träff et al., 2017). Davidse et al. (2014) report on two 9-year-old monozygotic twin girls, who had severe mathematical learning disabilities, but scored above average on word reading tests. They were not able to learn even basic mathematical skills. Their performance on a series of number sense skills was investigated and compared to a reference group of 8 age-matched girls. These girls scored significantly lower on all number sense tasks (numberline 0-10, magnitude comparison 1-16, and subitizing 1-4). They even scored at chance level on the comparison and subitizing tasks. In addition to their number sense deficits, these girls also showed poor working memory performance and poor visual-spatial skills, as well as poor spelling performance.

Träff and colleagues (2017) administered a comprehensive cognitive test battery to four children (two boys, two girls) with MLD aged 8-9 years old. Perhaps the most interesting finding was the heterogeneity in the profiles of the four children. Two of them showed number sense deficits and domain-general deficits (especially visual-spatial working memory). One only showed problems with the symbolic number sense tasks, but not with the non-symbolic, in combination with a general cognitive deficit (visuospatial working memory and executive functions). And one of them had only general cognitive deficits (verbal working memory, executive functions). The authors concluded that MLD cannot be attributed to a single explanatory factor, but that a multiple deficit account should be applied.

Although it is difficult to compare these case studies, due to the differences between participants and methodologies (measures, constructs), the conclusion seems justified that these studies do not converge to one conclusion. Although in most of the described cases number sense was impaired, this was not found in all cases (De Visscher & Noël, 2013; one out of four cases in Träff et al., 2017). Furthermore, in most cases some domain general deficits were found, but again not in all. According to Träff et al. (2017), this can be explained by recognizing different subtypes within MLD. When considering these different case studies, it indeed seems obvious that large differences exist between different people affected by MLD, making one general description almost impossible. Distinguishing between subtypes could be an interesting alternative to describe the characteristics of MLD. The next section will discuss research that has focused on these possible subtypes of MLD.

The Search for MLD Subtypes

To explain the large heterogeneity within the group of children with MLD, some have argued that several subtypes of MLD exist. Probably the most common distinction is the tripartite that Geary (2004) conceptualized: 1) a procedural subtype: Difficulties with strategies and concepts involved in advanced mathematics, 2) a semantic subtype: Reduced accuracy and speed for arithmetic fact retrieval, and 3) a visuospatial subtype: Difficulties in visuospatial skills. These three subtypes are assumed to be different not only in their mathematical problems, but also in their underlying cognitive characteristics and developmental patterns. Desoete (2007) proposed a fourth subtype, in which children's numerical cognition is impaired. Karagiannakis et al. (2014) continued on these profiles and distinguish between deficits in (1) core number, (2) memory, (3) reasoning, and (4) visual-spatial. They also link these deficits to specific cognitive characteristics and mathematical outcomes. Other profiles have been suggested as well, for example by differentiating between various numerical representations (Moeller et al., 2012). Others have focused on the comorbidity with for example reading or motor disabilities to distinguish between profiles (e.g., Pieters et al., 2015; Szűcs, 2016).

However, the various subtypes described above are theoretical in nature, and only limited empirical evidence exists. Pieters et al. (2015) have identified two subgroups based on data-driven model-based clustering: They found evidence for the procedural and for the semantic subtype. Bartelet et al. (2014) distinguished six profiles based on numerical abilities, although it is remarkable that the mathematical performance of these profiles barely differed. Salvador et al. (2019) used cluster analysis on a mixed group of MLD and typically developing students and found two profiles with weak arithmetic skills: one with number sense problems and one with visual-spatial problems, but again the two subtypes performed similarly on arithmetic achievement, although the within-group variance was large. Szűcs (2016) focused on working memory. Based on a meta-analysis of 36 studies, he found one subtype with weak verbal working memory, this subtype is also related to reading problems, and another subtype with weak visuospatial working memory (without reading problems).

To conclude, some subgroups might indeed exist within the group of MLD, although the results vary over studies and further research is needed to find more converging evidence. However, the results also show that it is difficult to find distinct cognitive profiles that are related to specific mathematical abilities, and that even within subgroups there is still much variability.

A Critical Reflection on former MLD Research

Sofar, we reviewed the evidence from different types of research on MLD. Below we will elaborate on the conclusions and provide our explanations for the main findings. In addition, we will critically reflect on the methods used in former research and how this might have affected the results of these studies.

First of all, MLD is a heterogeneous concept. Terms such as mathematical learning disabilities, mathematical learning difficulties and dyscalculia are used, sometimes with distinguished meanings, but without consensus about the specific definitions. In general, all of these terms point to serious problems in mathematical abilities, mostly to specific math problems and often to intervention-resistant problems. However, these three criteria (seriousness, specificity and resistance) are not always applied in the same way. Different cut-off criteria are used, both in practice and in research. The specificity criterion (i.e. children should not have additional disabilities and at least average performance on other academic skills) is often used in research. However, because of the high comorbidity rates (e.g., around 20-25% for dyslexia, spelling problems and attention disorders; Capano et al., 2008; Moll et al., 2014), in practice this criterion is less usable. Applying such criteria in research selects relative homogeneous samples that are not representative of the population, with - consequently - possibly faulty conclusions about characteristics of MLD. In contrast, the resistance criterion is very important in educational and clinical practices, but often not applied in empirical research. However, it should also be noted that in a recent meta-analysis it was shown that these criteria do not seem to make a major difference for the conclusions about MLD characteristics (Kroesbergen et al., 2021). Nevertheless, the heterogeneity in definitions and selection criteria makes drawing conclusions based on empirical studies quite difficult.

Secondly, conclusions about cognitive characteristics of MLD are mostly based on group comparisons. It has indeed been found that groups of children with MLD score on average lower on a number of skills, compared to groups of typically developing peers. These skills include - but are not limited to - number sense, working memory, processing speed, phonological skills, attention, spatial skills, ordering/ order processing, inhibition, number-specific executive function, and logical/non-verbal reasoning. It should be noted that most studies have compared groups of children with MLD, selected on strict criteria, with control groups (Astle & Fletcher-Watson, 2020). Differences between groups are then taken as evidence for a specific cognitive profile in the MLD group. These groups are often based on an (arbitrary) cutoff point along the normal distribution, while the children performing below this cut-off point are not necessarily qualitatively different from those who scored above that criterion (Peters & Ansari, 2019). Mammarella and colleagues (2021) tested the hypothesis that children with MLD are at the end of a developmental continuum, visible in impairments in many cognitive skills rather than having a core deficit in basic number processing skills. Data from a large sample were compared to simulated data to investigate the diagnostic power of possible underlying factors. They indeed found that none of the measured factors exceeded the diagnostic power that could be derived via simulation from the dimensional characteristics of a population. Applying a dimensional approach to learning disabilities might therefore be more valid (Peters & Ansari, 2019; Szűcs, 2016; Zhao & Castellanos, 2016). The assumption in this approach is that there is no qualitative discontinuity in the distribution from low to high performers.

Another problem with the method of group comparisons is that only means between groups are compared, and it can be questioned whether all children within these groups can be characterized by such a cognitive profile. As far as reported, this often seems not to be the case. For example, Kroesbergen and Van Dijk (2015) showed that a group of children with MLD significantly differed from their same-age peers in terms of working memory as well as number sense. However, when considering the specific individuals within the MLD group, only 38% of these MLD children indeed scored low on both working memory and number sense (and 23% showed neither working memory nor number sense problems). So, in this case, the group description could only be used to correctly describe about one-third of the individuals within that group. It might therefore be hazardous to use group comparisons to draw conclusions about characteristics of MLD problems in individuals. Qualitative analyses of individuals with MLD might provide a more nuanced understanding of their disability (e.g., Lewis et al., 2020), although the generalizability of case studies is small, and the review of case studies described here only stresses the variability between individuals with MLD.

This relates to a third conclusion: The heterogeneity within the group of individuals with MLD is enormous. As described above, group means are not applicable to all individuals, and conclusions drawn from group comparisons cannot always explain individual characteristics. The large variability between individuals within groups is often not studied, in contrary, heterogeneity is more often approached as 'noise' that should be controlled for. Even when the within-group differences are studied, the same approach usually leads to grouping individuals, i.e. research into MLD subtypes. However, for these subgroups, the same criticisms hold as for more inclusive groups. Even when a data-driven approach was used to distinguish between subgroups, the variability within subgroups is large (e.g. Huijsmans et al., 2020; Szűcs, 2016). It would do more justice to reality to take this variability into account and to use heterogeneity as evidence that a simple, uniform explanation of MLD is not possible and should be replaced by other theoretical models that take the variability in both cognitive and mathematical skills into account. According to Pennington (2006), development occurs through an interconnected network of (cognitive) skills. The development of mathematical learning difficulties could therefore depend on a different profile of cognitive deficits for each child. In addition, next to cognitive deficits, cognitive strengths might function as compensatory mechanisms and should be considered as well. Studying unique profiles of cognitive weaknesses and relative strengths might enlarge our understanding of MLD (cf. Huijsmans et al., 2021; Koriakin et al., 2016; Lewis & Lynn, 2018a).

The fourth, and probably most important, conclusion is that the causes of MLD are still not (fully) understood. The empirical evidence does not point to a single or fixed combination of factors that are apparent in all children with MLD. Although some cognitive characteristics have been described that might play a role, such deficits do not always lead to lower math performance. For example, not even half of the children with specific cognitive deficits (e.g., deficits in number sense, working memory, or rapid naming) have mathematical learning difficulties (Huijsmans et al., 2021; Kroesbergen & Van Dijk, 2015). These findings seriously challenge the assumed causal relations between the cognitive deficits and MLD. Furthermore, although group comparison studies generally assume such causal relations, the direction of these relations if often not examined. Peng and Kievit (2020) show, based on a review of both longitudinal and intervention studies, that the relation between cognitive abilities and academic achievement could best be described by bidirectional relations. This has indeed also been found for the relations between number sense and mathematics (e.g., Elliot et al., 2019; Friso-van den Bos et al., 2015). The assumption of a core deficit thus seems outdated, although the simplicity of this model might have had a strong appeal to both researchers and practitioners (see also Astle & Fletcher-Watson, 2020).

In our opinion, the heterogeneity in definitions and selection criteria as described above, as well as the variability in individuals with MLD, clearly point to the underlying problem that no evidence exists that MLD is a disability that is qualitatively different from (extremely) low performance, because no specific causes leading to specific symptoms in MLD have been found. Consequently, it depends on the used definition which children are labeled with MLD or not, with arbitrary and undesirable dichotomization of children with less or more severe mathematical learning difficulties as a result. This way of categorizing children is undesirable, because such labels could have important consequences for the education and interventions children receive.

Variation in Mathematics Revisited: A Multidimensional Model of MLD

Following our conclusions, dominant frameworks that help us to understand individual differences in mathematics learning may be in need of revision. According to the Multiple Deficit Model (MDM; Pennington, 2006; see also left panel of Figure 1) neurodevelopmental disorders, such as learning disorders, can be better understood, by studying their underlying etiology (genes, environment, and gene x environment interactions), brain mechanisms, neuropsychology, and behavioral symptoms. Cognitive multiple deficit models have been most successful for reading and mathematical disorders (McGrath et al., 2019), but have been criticized as well because they do not account sufficiently for heterogeneity in learning disabilities. Indeed, research from Huijsmans and colleagues (2020) suggested that children with MLD should be regarded as individuals with unique profiles of strengths and weaknesses that explain the way they learn mathematics in a similar fashion as their typical developing peers. In a cross-sectional study, they investigated to what extent specific profile(s) of mathematics difficulties and associated cognitive skills could be identified in a sample of 281 fourth grade children, using latent profile analysis (LPA). The results showed that children with MLD could not be separated from (low-achieving) typically developing children based on their profile of mathematics performance alone: 34% of the whole sample was grouped together into one profile consisting of weak performance on arithmetic and mathematics. Additionally, contrasting the cognitive skills of children with MLD to those of typically developing children did not result in separate profiles either. They stress that although their data-driven approach yielded different subgroups, the heterogeneity within the identified subgroups was still large. We propose that various sets of cognitive strengths and weaknesses are related to a wide variety in mathematical profiles, as visualized in the right panel of Figure 1.

Another point of criticism regarding the use of the MDM-framework in its current form, is that it does not fully recognize specific cognitive strengths that may compensate for cognitive weaknesses (McGrath et al., 2019). As a result, such cognitive strengths are often overlooked in research nowadays. To illustrate, children with MLD but without reading problems (but with deficits in number sense skills) were able to (partly) compensate their lower scores on fact retrieval and mathematics when they had high rapid naming skills (Huijsmans et al., 2021). Although it should be acknowledged that groups were relatively small, these results point in the direction that the consequences of cognitive risk factors for mathematical difficulties might be reduced through compensatory protective factors. It thus seems unfeasible to think about mathematics performance as a singular cause-effect relation wherein one (or few) core deficit causes math difficulties. In addition, the unidirectional relations from cognitive to behavioral characteristics also is a simplistic representation of the complex interaction of factors involved (Peng & Kievit, 2020).

We therefore propose a new multidimensional approach to MLD, in which both strengths and weaknesses on an individual level are recognized (see Figure 1). The rationale for this model is that children with MLD do not show different patterns of (cognitive predictors of) math development compared with typically developing children (Peters & Ansari, 2019; Szűcs, 2016; Zhao & Castellanos, 2016). Therefore, neither mathematical profiles nor cognitive profiles appear to be suitable to divide children with weaker math performance into separate groups, but mathematical abilities should be regarded as a dimensional construct. Furthermore, not all children within a group show the same behavioral or cognitive difficulties (Huijsmans et al., 2020). Such a complex interaction between multiple cognitive and behavioral factors requires a multidimensional approach. Additionally, learning difficulties do not result from cognitive weaknesses alone, but co-exist with strengths in other cognitive skills. That is why a multidimensional model of MLD might be a better representation of reality than a multiple deficit model. This model recognizes both cognitive weaknesses and strengths as relevant cognitive processes in the interaction with neural and behavioral characteristics. In such a model, certain combinations of cognitive weaknesses can result in the development of a specific learning difficulty, but on the other hand, specific strengths may partly compensate for such difficulties as well. As a result, the child's mathematical profile (i.e., behavioral phenotype) may also have specific strengths and difficulties. As a case in point, Huijsmans and colleagues (2020) showed that some children with mathematical learning difficulties had problems in fact retrieval, but not in advanced math (i.e., geometry, fractions), whereas for other children it was the other way around. Furthermore, a child that has both reading and mathematical difficulties, may have less cognitive strengths to compensate their mathematical abilities because of overlapping cognitive factors that account for the development in both reading and mathematics (i.e., phonological processing skills). Our model allows for such unique combinations of strengths and/or weaknesses on a cognitive level that relate to unique strengths and/or weaknesses of the child's mathematical profile.

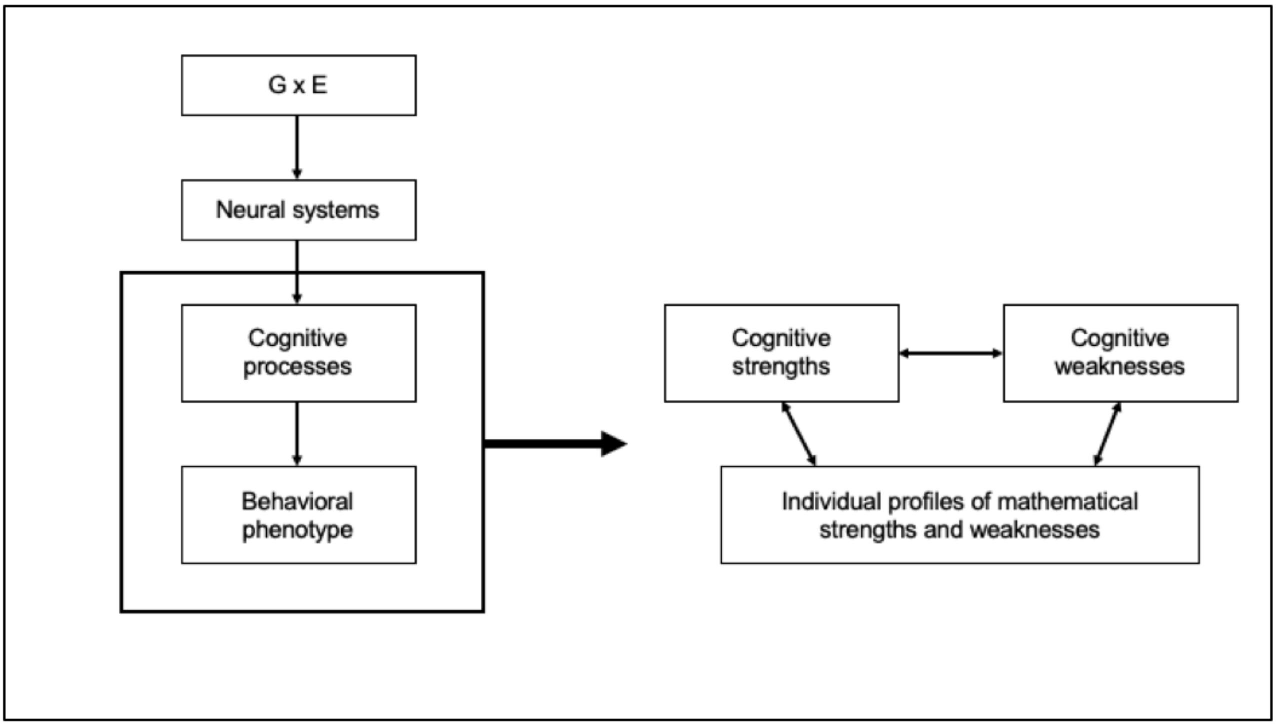
Of course, combinations of other cognitive strengths or weaknesses can be considered as well. For example, as both logical reasoning and working memory skills are needed to solve multi-component math problems (e.g., Kleemans & Segers, 2020) a child with relatively high logical reasoning skills may partly compensate for their relatively weak visual spatial working memory skills and, as a consequence, partly overcome their mathematical learning difficulties. Furthermore, as it turns out that affective processes such as a high self-efficacy can increase mathematical problem solving, despite having relatively weak working memory capacity (Hoffman & Schraw, 2009), factors including motivation and personality of the child, combined with domain-general and domain-specific cognitive factors should be considered as potential candidates for cognitive strengths/weaknesses as well.

To summarize, we view the development of disabilities such as MLD as a result of a unique combination of factors, fully recognizing both strengths and weaknesses, that impact on and work together in the process of learning a complex skill such as mathematics. As a consequence, mathematical learning difficulties should be seen as a system of causally connected symptoms rather than as effects of a fixed set of causal cognitive mechanisms. One of the challenges that needs to be addressed in future research is which specific combinations of strengths and weaknesses can account for individual differences in mathematical learning disabilities. Below we further elaborate on the implications this view has for both research and practice.

Implications for Research

A number of general implications for future research can be derived from our multidimensional model on individual differences in mathematical skills. First, the dichotomous definition of MLD (as opposed to non-MLD) should be reconsidered in scientific research. Commonly used methodological and statistical techniques aid differentiation between MLD and typical development as if they are two separate categories, but no evidence exists for such a qualitative difference. In addition, choices regarding sample selection are often ambiguous. This could have resulted in inconsistent conclusions across studies regarding the academic and cognitive profiles of MLD, and may have impeded the generalizability across empirical studies (Murphy et al., 2007). A more elegant perspective on developmental disabilities is the dimensional approach that views mathematics performance on a continuous scale (Hudziak et al., 2007; Moll et al., 2014), wherein some people perform somewhat better on this scale than others. The lowest range of weak performance on such a continuous scale would then be defined as MLD, preferably in interaction with one's profile of (cognitive) skills associated with the math difficulties. This dimensional approach to learning difficulties is different from a binary approach, because it is not based on one (set of) skill(s) or characteristic(s) that defines whether one has MLD or not. Instead, it does justice to the complexity of a skill such as mathematics by taking into account the large amount of individual variability within people.

Figure 1
Overview of the original Multiple Deficit Model by Pennington (2006) (left panel), and the elaborated multidimensional model (right panel). G x E = gene-environment interactions



When conducting research on learning difficulties, one should therefore be aware that although a definition like MLD can be used to describe the lowest-achieving group, not every individual within that group will have the same characteristics, because of the considerable amount of individual variation within the mechanisms associated with each child's math performance. In light of these reflections, future research on MLD ideally focuses on a variety of weaknesses but also strengths related to mathematics in line with our proposed multidimensional model, as well as environmental risk or protective factors. This approach will allow researchers to gain a comprehensive understanding of the development of MLD both at the group and the individual level (cf. Lewis & Lynn, 2018a; Mammarella et al., 2021)

As a consequence, alternative statistical methods, such as network analysis (Astle et al., 2019; Borsboom & Kramer, 2013; Fonseca-Pedrero, 2017; Zhao & Castellanos, 2016) may be a better way to fully account for intra- and interindividual differences in mathematics learning, next to the use of qualitative in-depth case studies (see e.g., Lewis & Lynn, 2018b) to better understand the cognitive strengths and weaknesses in mathematical learning on an individual level. Another option is to include larger variability in the samples: Not comparing selective groups (such as specific learning disabilities) but including participants with a range of mathematical - and other comorbid - problems (see also Astle & Fletcher-Watson, 2020). Only large samples enable research to find data-driven neurocognitive dimensions that might underlie learning problems (Astle et al., 2019) and improve statistical power and lower the risk of overestimating effect sizes (Mammarella et al., 2021).

Secondly, instead of investigating cognitive differences and similarities between MLD and typically developing samples, future research should elucidate how individual profiles are related to the differences and—more likely—the similarities in the educational needs of individual children. Given the fact that each child with or without MLD needs the same set of skills (with different degrees of reliance on each of the skills), it can be questioned how children can best be taught to become proficient in mathematics. Mapping a profile of an individual child's strengths and weaknesses in mathematics and related skills such as reading and cognition may seem promising in that respect, but is not easily integrated into treatment programs for MLD. More research is needed to find out how diagnostic criteria should be applied and when it is necessary to further investigate cognitive profiles. In addition, environmental factors such as education probably offer good potential to decrease differences between children's math skills. The emphasis of research on MLD could therefore not only focus on the identification of cognitive factors related to the differences between children, but on the tools that help a diversity of students to learn mathematics well. Potential questions in this regard could for instance be what type of instruction works best for those children; which (digital) methods aid the development of math skills; and which degree of differentiation is desirable. Research outside the classroom could further identify the elements that improve implicit learning at home; and how the school board and the nation's government can facilitate learning mathematics within schools.

Finally, the tension between desirability and feasibility of an individual variation perspective within primary schools should be considered in future research as well. Although such an approach would be desirable for all children, the question arises whether teachers and other educational professionals have sufficient resources (e.g., knowledge, time, and money) to implement an individual variation approach in the near future. Moreover, the question rises how desirable it is to regard every child within the classroom as an individual. Each child has to achieve the same curriculum-based goals at the end of primary education, so they must participate in instruction together as much as possible. Differences in the educational needs of children with weaker math performance as opposed to children with (above) average math performance probably are more quantitative in nature than they are qualitative. To elaborate, these children may need more instruction time, but what they are being taught should be unified. Future research should investigate how teachers can best be supported in employing differentiation in instruction to give each child the challenge and support they need. Other approaches, such as peer-assisted learning wherein stronger learners collaborate with weaker learners, have also shown promising results (Fuchs et al., 2019). In this way, policy makers and school principals can be assisted to make informed decisions about best practices on the implementation of an individual differences perspective.

Implications for Practice

Next to implications for research, some implications for clinical practice can be mentioned as well. To begin with, educational professionals in the field of primary school mathematics are recommended to move away from their existing frameworks of learning that views the worst performance on a continuous scale as a learning disability (e.g., a discrete group that is intrinsically different from children that belong to another group). Once a child has been identified as learning disabled, teacher expectations and learning goals are generally adjusted downwards for those children (Szumski & Karwowski, 2019). However, as it appears that the mathematical and cognitive profiles of children with weak math performance are quite similar to those of average to high achieving children, all children may most probably benefit from the same education within the classroom. The children at the lowest end of the continuum might need additional guidance and time in small, heterogeneous groups regarding topics they do not master yet. This could entail increased practicing with automatizing arithmetic operations, or systematically writing down intermediate steps when solving complex math problems (Gelderblom, 2010; Ruijssenaars et al., 2021).

Next to the fact that there are similarities in the educational needs of children with MLD, it should be noted, however, that evidence from various case studies suggest that large differences do exist between children, making one general description for what is needed to remediate their difficulties almost impossible. As a consequence, when mathematical learning difficulties are more severe and persistent, educational professionals are recommended to first carefully map the cognitive strengths and weaknesses of the child and then make adaptations to the educational context to match their educational needs. By observing children during math instruction, by examining patterns of errors within children's math work, and by discussing the strategies children use to solve math problems, teachers or other professionals could identify the cognitive strengths and weaknesses of the child. For instance, a child that cannot seem to remember intermediate steps or intermediate answers might have difficulties with his working memory, and a child that does not seem to grasp how to work with a number line might have problems with his number sense, and vice versa. Only when their unique profiles of weaknesses and strengths are being fully acknowledged, all children (including those at the higher end of the mathematics continuum) are able to receive high-quality education and will ultimately have the potential to meet their countries' national requirements for mathematics (Vaughn, & Fuchs, 2003).

Furthermore, the way MLD is currently being diagnosed in clinical practice appears to be somewhat ambiguous. Diagnostic criteria have been described in widespread manuals (DSM-5, APA, 2013; ICD-11, WHO, 2018), but these are based on a descriptive behavioral pattern only: Severe, persistent, and specific difficulties with learning mathematics. This descriptive diagnosis does not indicate possible causes that may have induced the learning problem for individuals with MLD, and discrepancies in the definition of MLD between research and clinical practice exist. The scientific basis for the way MLD is currently being diagnosed is quite weak (Peters & Ansari, 2019), and does not sufficiently differentiate between children with and without mathematical learning disabilities. As a result, this may have hindered the development of successful prevention and remediation programs for clinical practice. Abilities related to mathematics such as reading and cognition should not be overlooked in clinical practice either, and it is therefore advised that interventions for math difficulties become available for all children with mathematical difficulties, ranging from mild to serious. Furthermore, the interventions should emphasize a broad spectrum of strengths and weaknesses related to mathematics, again in line with our proposed multidimensional model.

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Screening for Characteristics of Dyscalculia: Identifying Unconventional Fraction Understandings

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Abstract

Researchers intending to identify the unique characteristics of dyscalculia rely upon the problematic and imprecise proxy of low mathematics achievement. Although detailed case studies of adults with dyscalculia have offered insight into its characteristics, we do not yet know if these characteristics are unique to dyscalculia and could be used to screen younger students for these understandings. To address this, we designed a group-administered written assessment based on the unconventional understandings found in adults with dyscalculia to investigate whether these understandings are atypical. In study 1, we assessed 390 grade 6-8 students to investigate the prevalence of these understandings. In study 2, we assessed 80 grade 6-8 students and recruited three students who demonstrated high levels of unconventional understandings. We collected additional assessment data and determined that all three students met stringent clinical dyscalculia criteria. These studies provide a proof-of-concept for designing dyscalculia screeners based on the characteristics identified in adults with dyscalculia.

Keywords:

Math, Learning Disability, Rational Numbers, Assessment

Introduction

Dyscalculia is a cognitive difference in numerical processing that results in persistent and significant problems learning even the most basic mathematics (Butterworth, 2005; Mussolin et al., 2010). It is estimated that approximately 6-8% of school-aged children have dyscalculia, also referred to as mathematics learning disability[1] (Gross-Tsur et al., 1996; Shalev, 2007). Unfortunately, research on dyscalculia has been hindered because of the lack of a validated and reliable assessment to identify students with dyscalculia (e.g., Geary, 2004; Mazzocco, 2007; Price & Ansari, 2013). Researchers currently identify students with this disability by administering a standardized achievement test and selecting a cutoff threshold, below which students are considered to have dyscalculia[2] . There is great variability in the assessments used and the cutoffs selected (Lewis & Fisher., 2016; Price & Ansari, 2013) suggesting that researchers may not be studying one common phenomenon. Of greater concern is the use of low achievement as a proxy for dyscalculia because of the myriad reasons that students may perform at a "low" level on a test. The current identification approach used by researchers cannot differentiate low achievement due to dyscalculia from low achievement due to social, affective, environmental, or instructional factors. Indeed, the use of low achievement to identify students with dyscalculia has resulted in the over-representation of students of color, non-native english speakers, and students from low SES backgrounds in the dyscalculic group (Hanich et al., 2001, e.g., Compton et al., 2012). The findings of studies relying upon this kind of identification approach may reflect characteristics of low mathematics achievement rather than dyscalculia per se. This fundamentally limits the validity of these findings and the field's efforts to delineate the unique characteristics of this disability.

Although students with dyscalculia often do have low mathematics achievement, researchers need a more precise way of identifying students with this disability. The Diagnostic Statistical Manual, Fifth Edition (American Psychiatric Association, 2013; DSM-5) requires that environmental, economic, and instructional factors are ruled out before a dyscalculia diagnosis. Furthermore, the DSM-5 recommends a stricter low achievement criterion – the 7th percentile rather than the more commonly used 25th percentile (see Lewis & Fisher, 2016 for a review). Unfortunately, research on dyscalculia has not moved to adopt these more stringent criteria. This may be partially due to the fact that to allow for statistical comparisons, researchers must ensure that a sufficient number of students meet the study's dyscalculia criteria (e.g., Geary et al., 2000). This may also be due to the fact that differentiating cognitive and non-cognitive causes of low achievement is time consuming, methodologically challenging, and often requires longitudinal data collection (e.g., Mazzocco & Myers, 2003).

To address the need for a dyscalculia screener that does not rely upon low achievement, Butterworth (2003) developed a Dyscalculia Screener. This screener measures the student's speed and accuracy on simple arithmetic and rapid quantity comparisons thought to be associated with number sense (Dehaene, 2011). Unfortunately, researchers have found that this assessment misidentifies students (both false positives and false negatives) based on longitudinal data (Gifford & Rockliffe, 2012; Messenger et al., 2007) and therefore it has not been used in research on dyscalculia.

Because the characteristics of dyscalculia are not yet understood, it remains unclear what measures a dyscalculia assessment should contain (Price & Ansari, 2013). As researchers attempt to identify and define the core characteristics of this disability (Butterworth, 2005), they are doing so with the imprecise criterion of low mathematics achievement. Reliance upon the problematic proxy of low achievement leads to "findings that are difficult to interpret, replicate, and generalize" (Lyon, 1995, p. 7). We argue that accurate identification of students with dyscalculia is the central challenge in this field.

To make progress in understanding the unique characteristics of dyscalculia and improve identification methods, researchers must take a radically different approach. Rather than starting with large samples of students identified with the imprecise proxy of low achievement, it may be more advantageous to start with small samples of extreme cases, as has been productive in defining other disabilities. By "extreme cases," we mean instances in which an individual's physiology or behavior is not aligned with structural or societal expectations and thus it appears to warrant categorization and classification[3]. Detailed study of extreme cases has been essential to identify the defining characteristics of other disability categories, including attention deficit hyperactive disorder (Lange et al., 2010), autism (Wolff, 2004; Verhoeff, 2013) and dyslexia (Duane, 1979). For each of these disabilities, early clinical identification of extreme cases led to defining characteristics of the disability that were used to identify and further refine the definition (e.g., Verhoeff, 2013). For dyscalculia, extreme cases could be adults with a long history of significant and pervasive issues with math (e.g., Mejias et al., 2012), who continue to struggle with arithmetic despite sufficient educational opportunities. Detailed analyses of these kinds of extreme cases can allow researchers to identify characteristic patterns of understandings evident in individuals with dyscalculia. Longitudinal studies have suggested that the difficulties experienced by students with dyscalculia persist over years (e.g., Lewis, 2014; 2017; Mazzocco et al., 2013), suggesting what is learned from adults with dyscalculia could inform investigations with younger students.

In this paper we draw upon characteristics identified in adults with dyscalculia in Lewis's (2014) case study work and design a pencil-and-paper assessment to investigate whether it is possible to identify these understandings in younger students on a group administered written assessment. We designed the written assessment based upon the Lewis (2014) case study for several reasons. First, this is one of the few detailed analyses of extreme cases of dyscalculia – focusing on basic fraction understanding for two adult students. Second, this study used a multidimensional identification approach (see Fletcher et al., 2007) which involved ruling out social and environmental causes for the students' low mathematics achievement, in addition to establishing that these students did not benefit from 1-on-1 tutoring instruction that was effective for younger typically achieving students. Third, common patterns of understandings were identified between the two students with dyscalculia, which were found to provide a productive explanatory frame for unexplained patterns of errors found in longitudinal studies of students with dyscalculia (Lewis, 2016; Mazzocco, et al. 2013). Fourth, these patterns of understandings were evident even in a student with dyscalculia who learned how to compensate effectively (Lewis & Lynn, 2018). Due to the persistence of these patterns of understanding and the commonality across students with dyscalculia, in this study we sought to evaluate how common these patterns were in students in general. The idea being that if these understandings were common in students with dyscalculia, but not typically achieving students, then these understandings could be used to selectively screen students for more extensive assessment and evaluation. The goal is to begin to disrupt the tautological relationship of low achievement and dyscalculia in the field, by identifying behavioral characteristics of the disability itself.

In this section we begin by presenting our sociocultural theoretical framing of dyscalculia, drawing upon Vygotsky's (1929/1993) conception of disability as qualitative human variation. We then describe the patterns of understanding identified in Lewis (2014), and consider these patterns in light of our theoretical framing.

Difference Not Deficit

Vygotsky's theory of disability is focused on understanding qualitative differences and is situated within his general theory of human development. Vygotsky (1981) argued that all human development progresses along two lines: the biological and sociocultural. For typically developing individuals, these two lines of development intersect. The individual's biological development intersects with the sociocultural line of development through social interactions which are mediated by tools (e.g., pencil) and signs (e.g., language). For individuals with disabilities, the sociocultural tools and signs that have developed over the course of human history may be incompatible with the individual's biological development (Vygotsky, 1929/1993). For example, spoken language is not accessible to a Deaf child and therefore does not serve the same mediational role to support the child's development of language as it would for a hearing child. In the case of students with dyscalculia, standard mathematical mediational tools (e.g., numerals, representations) may be incompatible with how these students process numerical information. Vygotsky (1929/1993) argued that this divergence of the sociocultural and biological lines of development does not result in an individual that is less developed, but an individual who has developed differently. This theoretical framing suggests that students with dyscalculia may use and understand standard mediational tools and signs in ways that are qualitatively different from and inconsistent with canonical mathematical usage. Therefore, analytically it is critical to attend to the unconventional ways that students understand and use standard mathematical representations.

Unconventional Understandings Identified in Fractions

Lewis (2014) identified unconventional fraction understandings in two extreme cases of dyscalculia – two adult students (ages 18 and 19). Both students entered their schooling with considerable privilege, both students were White, upper-middle class, and native English speakers. They attended well-resourced schools and both students had access to additional support and tutoring outside of school. Despite these supports, both students had low mathematics achievement and a long history of difficulties with mathematics which could not be explained by affective or environmental factors. These students also did not benefit from a series of tutoring sessions that were effective for younger typically achieving students (see Lewis, 2014 for details). A detailed analysis of video data from the tutoring sessions on fractions identified a small set of reoccurring and persistent understandings that the students relied upon, which were ultimately detrimental to their learning. These understandings involved using mathematical representations in unconventional ways. Both students had similar unconventional understandings which resulted in a similar pattern of errors. These unconventional understandings involved how students represented and understood the fraction ½ (halving understanding) and how they interpreted fraction representations in terms of the fractional complement (fractional complement understanding).

Unconventional halving understanding

The unconventional halving understanding involved representing the fraction ½ by halving a shape, in which the partition line itself was understood as the representation of ½ rather than 1 of the 2 parts (see Figure 1). For example, when students were asked to draw a picture of ½ they would draw a shape and partition it into two parts. When asked what part of their drawing represented ½, they would point to the partition line itself, often accompanying their explanation with a chopping gesture. Characteristic of this kind of understanding is a focus on the equality or balance between the two parts. For these students ½ was understood as an action, splitting, rather than a fractional quantity (e.g., 1 part out of 2). Although students' experiences splitting, partitioning, and sharing have been shown to be a productive resource upon which students can build (e.g., Empson, 1999; Steffe, 2010; Wilkins & Norton, 2011) the halving understanding was detrimental for both students in

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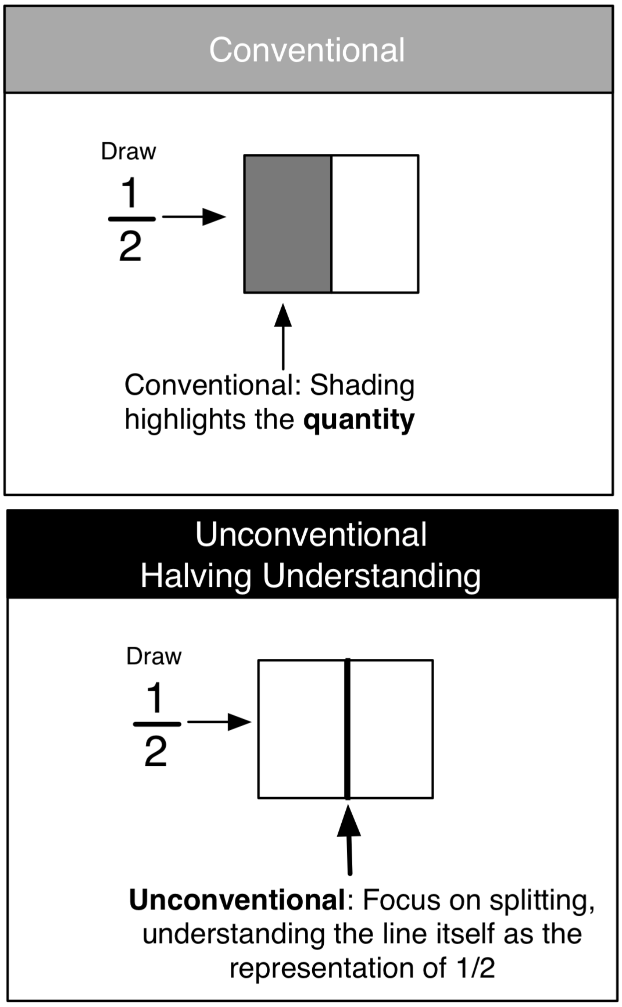
that it led to errors and limited the utility of various fraction representations (e.g., area models). Both students understood the fraction ½ as a process, rather than an object (Sfard, 1991), meaning that ½, the most intuitive and best understood fraction (Hunting & Davis, 1991) was not understood as a quantity.

Figure 1

Illustration contrasting the conventional understanding of one-half with the unconventional halving understanding found in students with dyscalculia (Lewis, 2014). Adapted from Difference Not Deficit: Reconceptualizing Mathematical Learning Disabilities, copyright 2014, by the National Council of Teachers of Mathematics. All rights reserved

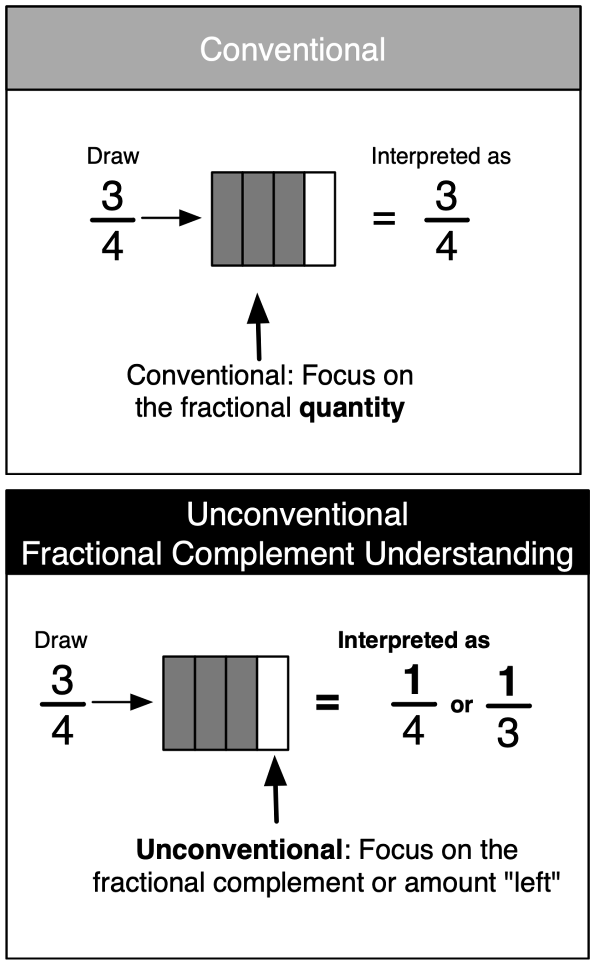


Unconventional fractional complement understanding.

The unconventional fractional complement understanding involved interpreting fraction representations in terms of the fractional complement. For example, interpreting an area model representation of ¾ as ¼ (unshaded/total) or 1/3 (unshaded/shaded), where the unshaded region was understood to be focal (see Figure 2)[4]. Although on the surface this might seem to be an issue of convention – attending to the white rather than shaded parts – for these students it reflected a disconnection between how students constructed and interpreted fractions. For example, when asked to draw the fraction ¾, they would draw a shape, partition it into 4 equal parts, and shade 3 of those parts. However, when asked what their own drawing represented, they would say "one-fourth" explaining that three parts were taken away, and one part was left. This suggested that these students did not have a stable way of representing a fractional quantity and the quantity itself transformed through the act of representing it. Characteristic of this understanding was conceptualizing the shaded fractional quantity as "taken away" or "gone" and referring to the unshaded fractional complement as an amount "left." More telling was that instructional attempts to correct this apparent "mistake" were not successful, even though the students knew that they made these errors, they could not stop themselves from thinking of the shaded as "gone" and the unshaded as "left" (Lewis, 2017).

Figure 2

Illustration contrasting the conventional understanding of area models with the unconventional fractional complement understanding found in students with dyscalculia (Lewis, 2014). Adapted from Difference Not Deficit: Reconceptualizing Mathematical Learning Disabilities, copyright 2014, by the National Council of Teachers of Mathematics. All rights reserved



Issues of access

These unconventional understandings (halving and fractional complement) were evident across a range of different problem types and representations. These understandings appeared when students were working with number lines, concrete fraction representations, and drawn pictures (e.g., area models). These unconventional understandings led to errors and resisted all standard instructional efforts to address them. These understandings were also not evident in typically achieving students who participated in the tutoring sessions. These halving and fractional complement understandings involved an issue of access, where standard mediational tools (e.g., fraction notation, area models) were not serving the purposes they were intended to support. Rather than understanding representations of fractions to show quantity, they understood these representations to show action (e.g., "taking"). Their understandings were, therefore, incommensurate with conventional mathematics use. Perhaps because the students understood fractional quantities as processes rather than objects, they had difficulty using these fractional quantities in other processes (e.g., adding ½ and 1/3 or finding an equivalent fraction for ¾) (Sfard, 1991). Not only did the unconventional understandings persist through the weekly tutoring sessions, but follow up studies suggested that these understandings persisted across multiple years (Lewis, 2017).

The Current Studies

To evaluate the prevalence of these kinds of understandings and the utility of using these characteristics to screen students, we designed a 13-item group administered paper-and-pencil assessment. We refer to this assessment as a "Screener" because we are specifically interested in screening students for halving and fractional complement unconventional understandings. The screener questions were based on questions from Lewis (2014) in which students demonstrated these unconventional understandings. Students were asked to draw, interpret, compare and operate with a variety of fractional quantities. For a complete list of questions with scoring guide see Appendix A. The screener questions were deliberately designed to elicit evidence of halving or fractional complement understandings, therefore, we did not specify the manner in which students should interpret fraction representations. Students were given one unconventional understanding point for every problem in which their answer reflected a halving or fractional complement understanding. A higher score on the screener meant the student demonstrated higher levels of unconventional understandings.

We evaluated the promise of this kind of screener with two studies. In the first study we evaluated how common these patterns of understanding were in a large sample (*n* = 390) of middle school students (i.e., grades 6-8; ages 11-14). Study 1 addressed the following research questions:

1. *Can unconventional fraction understandings (halving and fractional complement) be identified on a group administered written assessment?*
2. *What is the prevalence of these kinds of understandings?*
3. *Are unconventional understanding scores correlated with mathematics achievement scores?*

In the second study, we used this assessment to selectively recruit students to participate in an individual interview and assessment to determine whether students who demonstrated these unconventional understandings met rigorous DSM-5 dyscalculia criteria. Study 2 addressed the following research questions:

1. *Do students with high unconventionality scores on the Screener demonstrate the same unconventional understandings during a clinical interview?*
2. *Do these students with high unconventionality scores meet rigorous DSM-5 dyscalculia criteria?*

These studies together establish that building off the unconventional understandings identified in detailed analyses of extreme cases provides alternative avenues to selectively screen for characteristics of dyscalculia.

Study 1

In Study 1 we sought to evaluate whether it was possible to use the group administered Screener to identify the characteristic understandings found in students with dyscalculia. This paper-and-pencil assessment (see Appendix A) was administered to 390 students in grades 6-8 (i.e., middle school students, approximate age 11-14). Middle school (grades 6-8) was selected as the target age because these students would have had adequate exposure to fractions, given that fractions instruction generally begins in grade 3 in the United States (e.g., Common Core State Standards for Mathematics; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). We also collected state mandated achievement test mathematics scores to evaluate whether unconventionality scores were inversely correlated with achievement.

Methods

Data Collection

Mathematics teachers (*n* = 6) at a California middle school administered the Screener to all students during math class (*n* = 390). The teachers also provided each student's state mandated achievement test mathematics score from the prior academic year. In

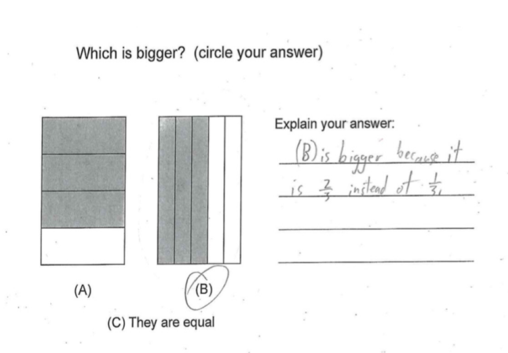
California, at the time, the mandated achievement test was the STAR test (Standardized Testing and Reporting program; http://www.cde.ca.gov/ta/tg/sr/). In order to collect these data and preserve student anonymity (a stipulation of our human subjects approval), when students completed the Screener, the teacher removed the cover page (with the student's name) and wrote the student's STAR test score on the now anonymized assessment. The research team received anonymized written responses on the Screener along with the student's STAR test score. One out of six of the teachers did not provide STAR test scores for her 50 students.

Analysis

Our research team scored the screeners for correctness and evidence of unconventional understandings. We assigned one unconventional understanding point for each answer which was consistent with an unconventional halving or fractional complement understanding (see examples Appendix A). For example, (see Figure 3) the student interpreted an area model of 3/5 and 3/4 as 2/3 and 1/3 (unshaded parts/shaded parts), respectively. The student was given one unconventional point for this problem because the student's response (which treated the unshaded pieces as focal) aligned with a fractional complement understanding.

Figure 3

Student work "(B) is bigger because it is 2/3 instead of 1/3." This answer would receive one unconventional understanding point for fractional complement because 3/4 and 3/5 were interpreted in terms of the fractional complements, 1 unshaded part for ¾ and 2 unshaded parts for 3/5, respectively (i.e., 1/3 and 2/3; unshaded/shaded)



Reliability and Validity Measures

All assessments were scored by at least two different scorers (see Appendix A for scoring criteria) Reliability for scoring was high: 97.9%. All discrepancies were resolved during our research meetings by reviewing the students' answers and our scoring criteria and reaching a consensus decision.

To evaluate the validity of this screener we conducted an item factor analysis. The parallel analysis showed that there is more than one factor measured by the test. The exploratory factor analysis further confirmed that a two-factor model outperformed a one factor model for these data. We determined the two factors were, as hypothesized: halving and fractional complement. Items 3 and 4 were removed from the confirmatory factor analysis because they were not associated with either fractional complement or halving. The confirmatory factor analysis showed that Items 1, 2, 5, and 12 loaded on factor 1 (halving), with questions 1 and 2 ("draw ½" and "draw another way to show ½") loading strongly on factor 1 (halving). The confirmatory factor analysis indicated that items 6, 7, 8, 9, 10, 11, 13 all strongly loaded onto factor 2 (fractional complement). The results of the confirmatory factor analysis are presented in Table 1, standardized factor loadings are between -1 and 1, with larger absolute values indicating a stronger association between the item and the factor. Because this screener is measuring two factors, Cronbach's alpha was understandably low (0.61), but the correlation between the two factors was moderately high (0.31).

Table 1

Standardized Loadings for 2-Factor Confirmatory Model of Unconventional Fraction Understandings (n = 390)

Item Number	Question Description	Factor 1 – Halving	Factor 2 – Fractional Complement
1	Draw ½	0.98	
2	Draw ½	0.99	
5	Interpret ½	0.35	
6	Compare 1/6 and 1/8		0.82
7	Compare 2/8 and 5/8		0.74
8	Interpret area model of 4/5		0.79
9	Interpret area model of 8/10		0.76
10	Compare ¾ and 3/5 area models		0.86
11	Compare 4/5 and 3/5 area models		0.90
12	½+1/4 =	0.60	
13	Interpret eight 1/10		0.53

Results

The results for Study 1 are presented in three parts. First, to evaluate whether it was possible to identify unconventional understandings on a written assessment, we present some exemplar written responses which illustrate unconventionality, either a fractional complement or a halving understanding. Second, we present an overview of the students' scores on the Screener to report the prevalence of these understandings. Finally, we evaluate whether unconventionality scores on the screener were associated with students' standardized mathematics achievement test performance.

Exemplar Unconventional Understandings

To illustrate prototypical unconventional understandings, we present several examples from students' responses and discuss how these reflect a potential halving or fractional complement unconventional understanding.

Unconventional Halving Understanding

Students' answers were coded as consistent with a halving understanding (Lewis, 2014) if they drew or interpreted the fraction ½ as a halved shape (see Figure 4). For example, a "halving understanding" was reflected in Figure 4a because the student drew a shape and partitioned it in two but did not shade or label either piece. Similarly, instances in which students selected an unshaded halved circle as a valid representation of ½ were considered consistent with a halving understanding (see Figure 4b). Finally, some students represented the fraction ½ without shading when asked to solve the problem ½ + ¼=. In this particular example (see Figure 4c), the student represented both ½ and ¼ with no shading. It is unclear what the student's intermediate drawings were intended to represent, but their final answer (an unshaded halved shape) was interpreted as ½ in their final answer and therefore was coded as consistent with a halving understanding.

Unconventional Fractional Complement Understanding

The fractional complement understanding occurred more often in cases in which the problem involved interpretation of a fraction. For example, student answers indicative of a fractional complement understanding included judging eight one-tenth pieces to be 2/10 (pieces missing/total pieces; see Figure 5a) or 1/8 (one empty space/pieces shown; see Figure 5b). Similarly, answers in which the student interpreted an area model in terms of the unshaded pieces (e.g., interpreting 4/5 as 1/5 (unshaded/total) and 8/10 as 2/10 (unshaded/total)) were also coded as indicative of a fractional complement understanding (see Figure 5c).

Figure 4

Exemplar written responses coded as consistent with a halving understanding

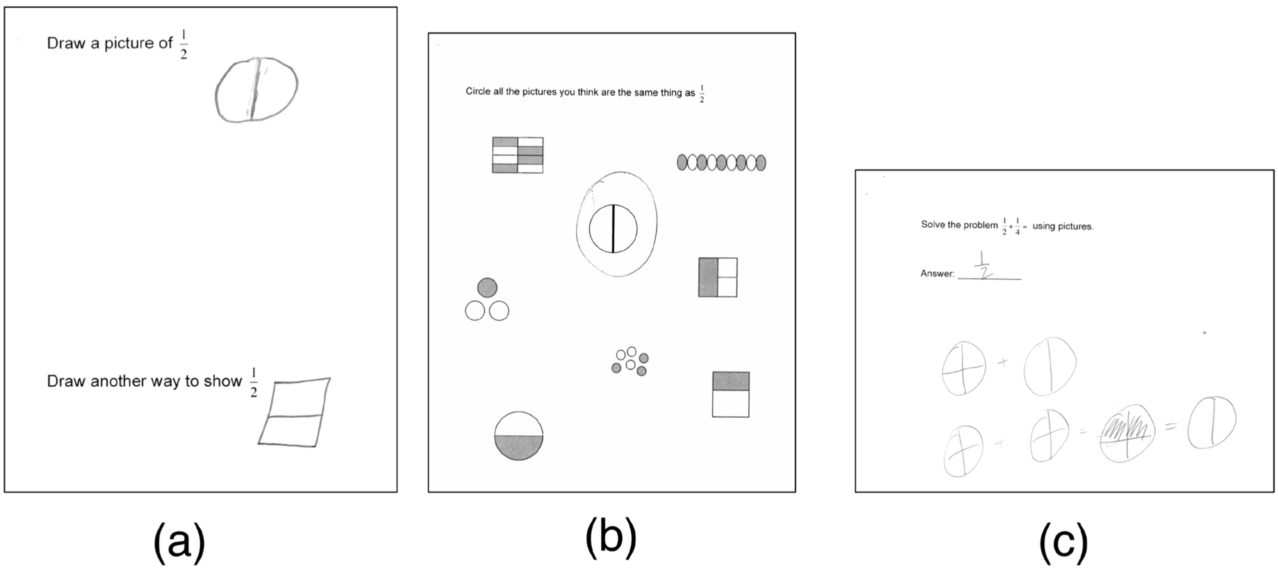
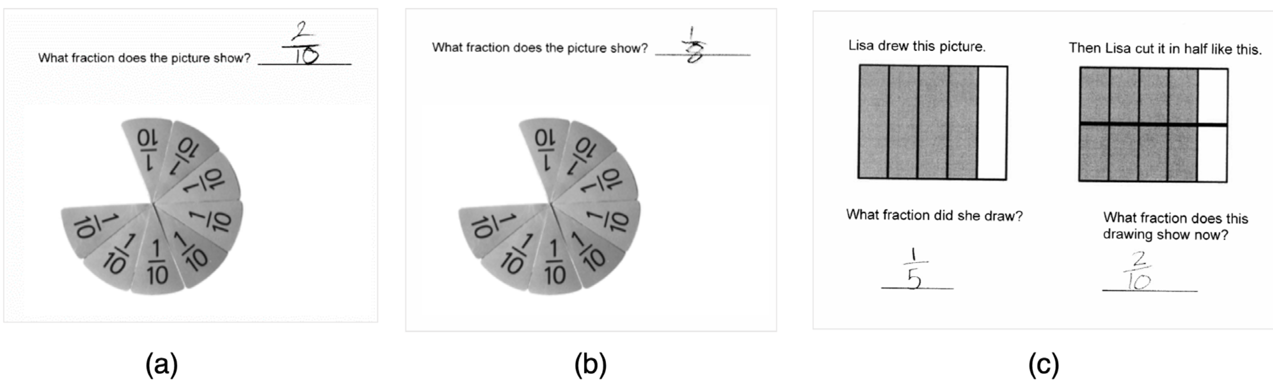


Figure 5

Example answers coded as consistent with a fractional complement understanding



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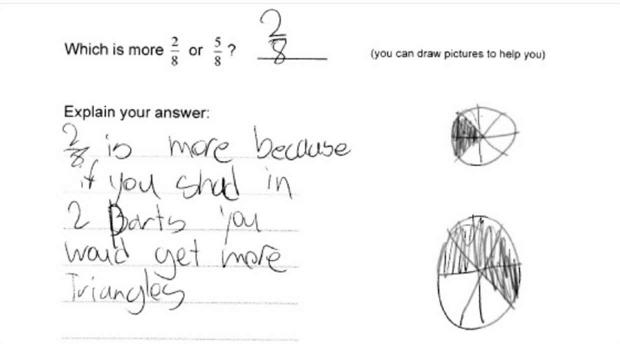
Fractional complement understanding was also evident in errors involving comparison of fractions. For example, a student incorrectly judged that 2/8 was larger than 5/8 explaining, "2/8 is more because if you shad in 2 parts you woud get more triangles" (see Figure 6). In this example, the student presumably sees more "triangles" in the drawing of 2/8 because there are 6 unshaded parts versus the 3 unshaded parts in the drawing of 5/8.

Figure 6

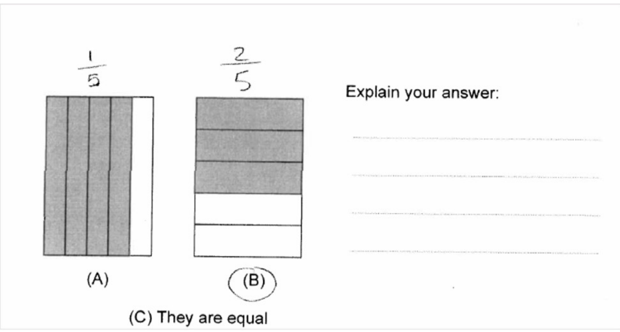
Student answer and explanation that 2/8 is more than 5/8



Similarly, when students made errors on comparing an area model of 4/5 and 3/5, their answers often reflected a fractional complement understanding. For example, one student interpreted the area model of 4/5 as 1/5 and the area model of 3/5 as 2/5 (see Figure 7). In both cases the student attended to the unshaded parts as the focal fractional quantity and therefore incorrectly determined that the latter was larger.

Figure 7

Student answer and explanation that an area model for 3/5 is larger than 4/5



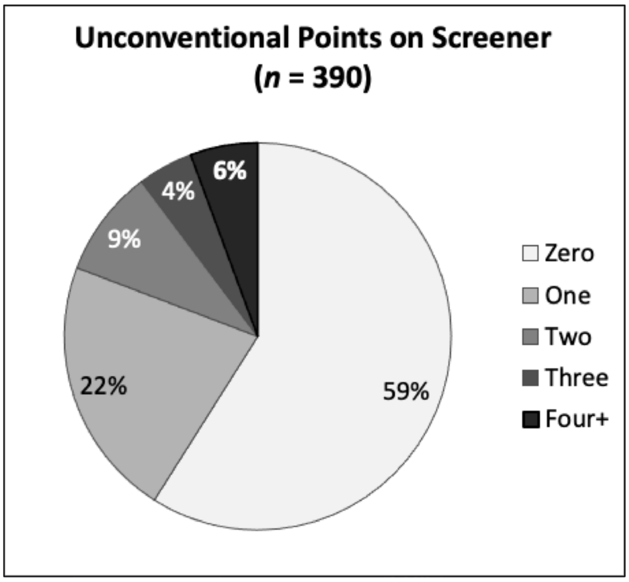
Although both halving and fractional complement are distinct understandings on this Screener we totaled the number of answers which were consistent with either a halving or a fractional complement understanding to produce one total score of unconventionality. A student receiving a higher unconventionality score would have more answers which indicated a fractional complement or halving understanding.

Student Performance on the Screener

To analyze the students' scores on the Screener, we considered the total unconventionality score obtained by each student. Only 6% of students had an unconventionality score of four or more points (indicating answers aligned with an unconventional understanding on more than 30% of problems; see Figure 8). The majority of students (59%) demonstrated no unconventional understandings. Another 22% of all students received only 1 unconventional point, and more than 63% of these students received an unconventionality point for circling the unshaded circle partitioned in 2 as one possible representation of ½ along with other valid representations. Therefore, as expected, most students demonstrated no unconventional understanding on the Screener.

Figure 8

Percentage of students who scored in each unconventional point range on the Screener (n = 390)



Achievement Test Scores

We collected student achievement scores to investigate whether unconventionality was simply a characteristic of low mathematics achievement (and consequently simply replicating achievement measures). We evaluated whether the students' unconventionality scores were correlated with their standardized achievement test scores. For this analysis we omitted 89 students for whom we did not receive STAR achievement mathematics test scores. One teacher did not provide this information to the research team (n = 50), and there were missing data for specific students in other classes. This missing data could be due to a variety of reasons, student's absence during STAR testing, transferring to the district or class, or an error of omission on the teacher's part.

When viewing the scores as a scatterplot (see Figure 9), it is evident that the students with the highest unconventionality scores were not necessarily the lowest achieving students, and some of the lowest achieving students had no unconventionality points. This suggests that this kind of approach – identifying characteristic patterns of reasoning – may be a promising approach to begin differentiating dyscalculia from low mathematics achievement due to other factors.

Figure 9

Scatterplot of achievement test scores and unconventionality points on the Screener, identical values are jittered

Summary and Conclusion

Study 1 found that the unconventional understandings documented in students with dyscalculia were evident on the group administered written Screener. This study suggests that these unconventional understandings, previously only documented with time intensive qualitative analysis of video data, are possible to identify in a group administered screener. Furthermore, the percentage of students with higher unconventionality scores (i.e., 4+ points) was approximately equivalent to the estimates for prevalence of dyscalculia (Shalev, 2007). Data from state mandated assessments suggested that high unconventionality scores were not only occurring in the lowest achieving students; furthermore, not all low achieving students demonstrated these unconventionalities. This suggests that this screener is measuring something different than low mathematics achievement. Due to the anonymized nature of the data we were not able to follow up with individual students who had high unconventionality scores. It remained an open question whether students who demonstrated high levels of unconventionality on the assessment would continue to exhibit these understandings over time and whether those students would also meet standard dyscalculia identification criteria. To investigate these questions, we conducted Study 2.

Study 2

In Study 2 we wanted to determine if the unconventional answers given on the Screener persisted and whether these students met rigorous DSM-5 dyscalculia criteria. We administered the Screener to 80 middle school students and recruited those students with high unconventionality scores to participate in an additional individualized assessment. The criteria for "high unconventionality" was set at four or more unconventional points, because this indicates reliance upon unconventional understandings across a significant number of problems (i.e., more than 30% of problems). Although it may have been interesting to assess students with two or more unconventional points to determine if they have an unconventional understanding of standard pedagogical representations, we focused on students with the highest levels of unconventionality (4 or more points) due to time constraints. We conducted individual problem solving clinical interviews to evaluate whether these students did rely upon unconventional understandings. We conducted an individualized standardized achievement test (Woodcock Johnson IV; Schrank et al., 2014) and background interview to determine whether these students with high unconventionality scores met standard DSM-5 dyscalculia criteria.

Methods

Data Collection

All middle school students (grades 6-8) enrolled at a private school for students with language-based learning disabilities were assessed using the Screener (*n* = 80). The student's enrollment at this school ensured that these students had intelligence scores in the normal range and therefore eliminated the possibility of intellectual disability. We anticipated that a higher percentage of students recruited from this school would have high unconventionality scores given the documented comorbidity between dyscalculia and dyslexia (e.g., Knopik et al., 1997; Wilson et al., 2015). However, Lyon, Shaywitz and Shaywitz (2003) argue that although there is well known comorbidity, the cognitive characteristics associated with each of these disabilities are sufficiently distinct (e.g., phonemic awareness vs. number processing) and do not present a problem in studying one independent from the other. In addition, the reading demands of the screener were minimal, and therefore, the impact of the student's difficulties with reading were not considered to be problematic for this study.

Each mathematics teacher at the school administered the Screener to their students (n = 80). The cover page and first page of the assessment were numbered with a test ID. When students completed the assessment, the teacher removed the cover page (with the student's name), and retained the cover sheet for subsequent recruitment efforts. The research team scored these assessments anonymously. To recruit students for the main study, the teachers were given a list of test IDs associated with students who had unconventionality scores of at least 4 points. Teachers used the cover sheets to distribute consent materials to students who qualified. Consents were directly returned to the research team through the U.S. Postal Service. Parents and students who did not want to participate were asked to simply discard their forms to preserve their anonymity. Seven students met the high unconventionality threshold and we received consent forms for three of these students.

Several kinds of data were collected for the three students who participated in the individual assessment including: (a) background interview, in which the students reported on their resources and their prior experiences learning and doing mathematics, (b) a clinical interview problem solving session in which the student solved the questions from the Screener, and (c) an individually administered standardized achievement test. Due to scheduling constraints these individual sessions were conducted eight months after the original assessment data.

Background interview

The students were interviewed and asked to provide a self-report of their academic background, the kinds of difficulties they experienced in mathematics, their level of effort, available resources (e.g., tutoring, teacher help), and home language (see Appendix B). The goal of the background interview was to assess the student's level of perceived effort and educational resources as well as to establish rapport. Note that we did not collect data on the socioeconomic status of the student and their families, but these students were all paying tuition to attend a private school, suggesting the families had sufficient financial resources.

Problem solving interview

In the problem solving clinical interview, the students were asked each of the questions from the Screener. For each of the student's answers, the interviewer asked the student to explain their solution and/or process. Because it had been over eight months between the administration of the Screener and the interview, we were not concerned about practice effects.

Both the background interview and problem solving interview were video recorded and were conducted by the first and second authors.

Standardized measure. To determine if the students met the low mathematics achievement clinical criteria established in the DSM-5, all three students were assessed using the mathematics subtests of the Woodcock Johnson IV Test of Achievement (Schrank et al., 2014). The subtests included, Applied Problems, Calculation, and Math Facts Fluency.

Analysis

Screener

As in Study 1, the written screener assessments were scored by at least two different scorers (see Appendix A for scoring criteria). Reliability for scoring was high, 97.6%. All discrepancies were resolved during our research meetings by reviewing the students' answers and reaching a consensus decision.

Case study analysis

For the three students who qualified for and consented to participate in the individual assessment, we transcribed the video recordings and scanned all written artifacts.

Background interview

For the background interview, the first and second authors reviewed the students' answers and identified any potential confounding factors which could explain the student's mathematics difficulties. We looked for self-reports of insufficient educational opportunity, insufficient resources, poor prior teaching, or difficulty with attention or behavioral control.

Problem solving interview

For the problem solving interview, the first and second authors coded these videos using the coding scheme from the Screener. We allowed student's explanations to disambiguate answers when needed, similar to the way we used written explanations on the Screener. Reliability for this coding was 85.2%. All discrepancies were resolved by reviewing the video and reaching consensus on how the question should be scored.

Results

The results are presented in two parts. First, we present each case by illustrating the students' unconventional answers from the screener and how these same patterns of reasoning were evident during the interview. Then we evaluate whether these three students met the standard DSM-5 dyscalculia criteria.

Case Study Students with High Unconventionality Scores

Out of the 80 students assessed, only 7 students (9%) had an unconventionality score of four or above. These 7 students were recruited to participate in the interviews and standardized assessment. Three students, "Ryan," "Lily," and "Maddie" (all pseudonyms) returned consent forms. All three students who consented to participate in the main study demonstrated the same unconventional understandings during the interview that they did on the screener (see Table 2). Ryan and Lily demonstrated halving and fractional complement understandings on both the screener and interview. Maddie demonstrated a fractional complement understanding, and did so on both the screener and interview. For each case study student, we present answers given on both the screener and the interview which highlight the persistence of these unconventional understandings.

Table 2

Unconventional understanding points on the screener and interview

	Assessment	Fractional Complement Points	Halving Points	Total Unconventional Understanding Points
Ryan	Screener	3	1	4
	Interview	1	3	4
Lily	Screener	3	1	4
	Interview	6	1	7
Maddie	Screener	4	0	4
	Interview	5	0	5

Ryan

On the screener Ryan demonstrated both an unconventional halving and fractional complement understanding. In Ryan's answers on the screener, a halving understanding was evident on one problem, in his selection of the non-shaded halved circle as a valid representation of ½. Ryan also demonstrated a fractional complement understanding in his comparison of fractions on the screener. When asked to compare fractions, he incorrectly judged 1/8 to be greater than 1/6 and 2/8 to be greater than 5/8 drawing accurate areas models for each. He also incorrectly judged an area model for 3/5 to be greater than 3/4, and an area model of 3/5 to be greater than 4/5. In each instance, his explanations identified "more space" in the fraction he judged to be larger, which was consistently the fraction with more unshaded parts. This suggests that, particularly on comparison problems, Ryan was relying upon a fractional complement understanding.

On the interview both these unconventional understandings resurfaced but with different frequency. A halving understanding occurred more frequently, and fractional complement understanding occurred less frequently. When Ryan was asked to draw the fraction 1/2, he drew several different representations including a pizza, a pie, and a pedestrian "don't walk" sign (see Figure 10). In each of these cases, he omitted shading. When asked to identify the part of his picture that was one-half he indicated that one-half was the partition line.

Figure 10

Ryan's drawings of 1/2 (pedestrian "don't walk" sign, pizza, and pie)

Interviewer: Can you explain to me how your pictures show one-half?

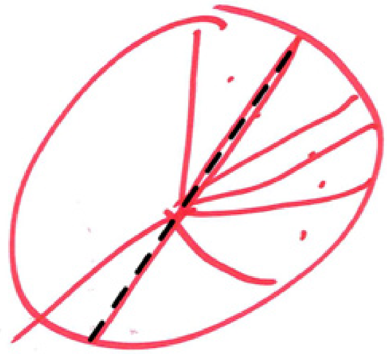
Ryan: Um, because they have a line right down the middle [points to line in the center of the pie, see Figure 10], and this side's equal [points to right side of pie], and this side's equal [points to left side of pie]. Like 1, 2 [writes ½] or... [starts pointing to the pizza slices in his drawing] I don't know how many pieces of pizza that is, but, yeah.

Interviewer: So where is the one-half in this picture? [points to pizza]

Ryan: [points along center dividing line; see Figure 11] Right there.

Figure 11

Ryan's drawing of 1/2 of a pizza with a dotted line indicating where he gestured to identify where one-half was in his drawing



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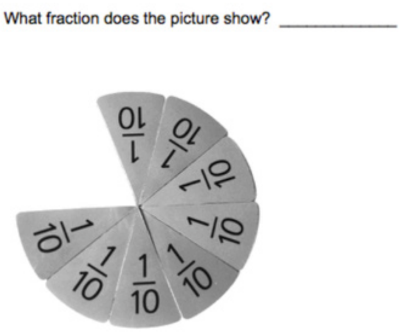
In Ryan's explanations he focused on the equality of the two halves and the partition line itself. Although Ryan's drawing of the pizza pieces and his attempt to count them up, suggests that he might have been attending to one-half of his circle (or pizza), when specifically asked where the one-half was in his picture, he identified the partition line itself and not the pieces on one side of the pizza as the representation of 1/2. Ryan's unshaded and halved representations along with his explanations focusing on the partition line itself was taken as evidence of his halving understanding.

In contrast to Ryan's halving understanding, a fractional complement understanding occurred only once during his interview. On an interpretation problem, Ryan determined that the eight 1/10 pieces (see Figure 12) was equal to 1/8. This reflected a fractional complement understanding because he attended to the missing part (perceived as 1 missing part) and the number of pieces displayed (i.e., 8). This was coded as a fractional complement understanding because it involves naming the fraction in terms of the missing amount.

Figure 12
Interpretation problem which presents eight 1/10 pieces and asks student to interpret the amount shown

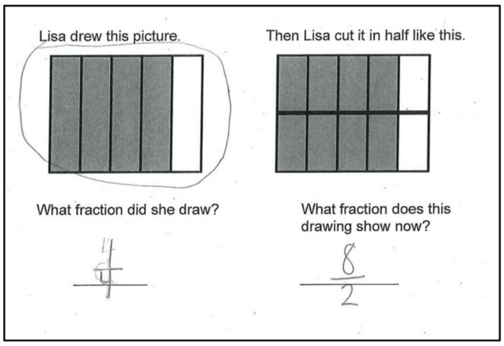


Although the halving and fractional complement understandings were evident on different problems and had different frequencies on the screener and in the interview, in both instances, Ryan's answers and explanations indicated his reliance upon these understandings found in students with dyscalculia.

Lily

Lily demonstrated both a halving and fractional complement understanding on the screener and interview. Like Ryan, Lily selected the unshaded halved circle as a valid representation of ½, and did so both on the screener and interview. Therefore, there was consistency in her halving understanding. Lily also demonstrated consistency in her fractional complement understanding. On the screener Lily interpreted 4/5 and 8/10 as 1/4 (unshaded/shaded) and 8/2 (shaded/unshaded), clearly attending to the unshaded pieces as focal (see Figure 13). In addition, many of her area model comparison problems were also aligned with attending to the fractional complement (e.g., larger fraction determined by largest unshaded area; Figure 14), but these were not coded as such because she did not provide a written explanation for her judgments.

Figure 13
Lily's screener responses that were coded as consistent with a fractional complement understanding, because she focused on the unshaded (fractional complement) pieces in her interpretation of the fraction



Lily's interpretation of the area models 4/5 and 8/10 during the interview was similar to her answers on the screener. During Lily's interview, she again identified 4/5 as 1/4 (unshaded/shaded) and identified 8/10 as 2/8 (unshaded/shaded), focusing on the pieces she referred to as "left."

Figure 14
Lily's answers that were potentially due to a fractional complement understand (judging fractions based on unshaded parts) but were not coded as unconventional, because she did not provide an explanation for her answer

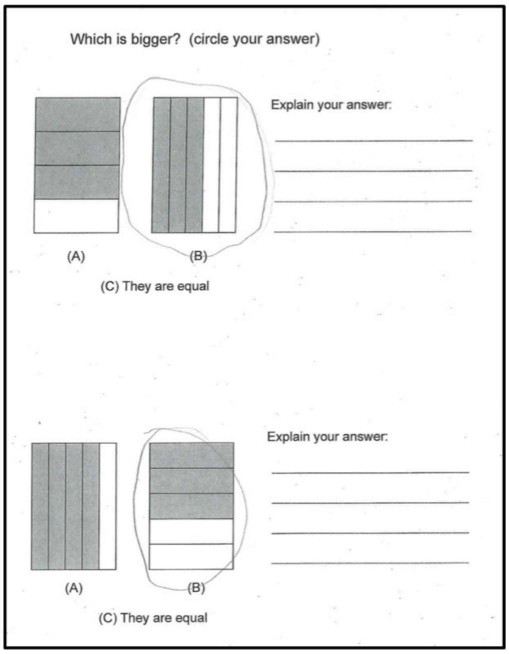
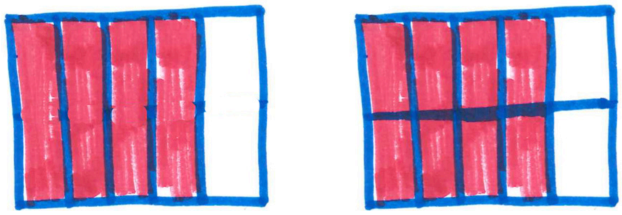


Figure 15

Tutor drawn representation of 4/5 (digitally recreated), which was repartitioned to produce 8/10



(a) (b)

Interviewer: [draws 4/5; see Figure 15a] Okay, so this is a picture of –

Lily: One-fourth.

Interviewer: So this is a picture of one-fourth?

Lily: Yeah.

Interviewer: Okay. So then another student came along and did this to her picture. [draws horizontal line; see Figure 15b] Can you tell me what fraction that is?

Lily: [pointing to unshaded sections] Is she crossing out this? Oh.

Interviewer: So she...

Lily: Two-eighths.

Interviewer: Two-eighths?

Lily: Yeah.

Interviewer: Okay. Can you tell me how you got that answer?

Lily: Well, [points to picture], if you divide it in half, this makes 8, because 1, 2, 3, 4, 5, 6, 7, 8, [gestures over 8 shaded pieces, each in turn] and then there's 2 left over [points to 2 unshaded pieces].

Lily interpreted the fraction in terms of the unshaded pieces and referred to those pieces as "left." Lily's tendency to interpret fractions by attending to the fractional complement (unshaded parts) also emerged as she compared area models of 3/4 and 3/5. As she had done on the screener, she judged the drawing of 3/5 to be larger. When asked to explain her answer, she interpreted each fraction in terms of the number of unshaded parts and shaded parts; 3/5 was interpreted as 2/3 and 3/4 was interpreted as 1/3.

Interviewer: In looking at these two pictures, can you tell me which one is larger, or are they equal?

Lily: [touches drawing of 3/5 firmly with finger, 5 times; see Figure 16] This one.

Interviewer: Do you want to circle it?

Lily: Naw, that's okay. Just that one [points to drawing of 3/5].

Interviewer: Can you tell me – you're pointing to this one –

Lily: Yeah.

Interviewer: – it's larger? Can you tell me how you know that?

Lily: [points to drawing of 3/5] There's... it's two-thirds, and then this one is [pointing to drawing of 3/4], one-third. So this one's more [points to drawing of 3/5], there's 2 that got left out kind of.

Lily's judgment that 3/5 was larger than 3/4 was based on her attention to the unshaded pieces, which she again referred to as "left out." Lily consistently relied upon a fractional complement understanding. Given the consistency of Lily's answers on both the screener and the interview, the fractional complement understanding provides a plausible explanation for Lily's errors on the area model comparison problems on the screener (see Figure 14).

Figure 16

Printed question asking student to compare ¾ and 3/5 represented with area models

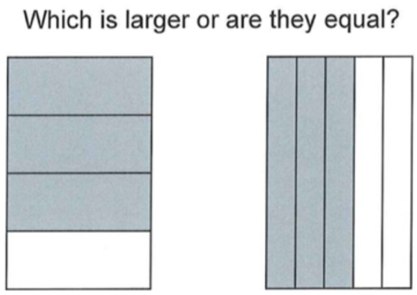
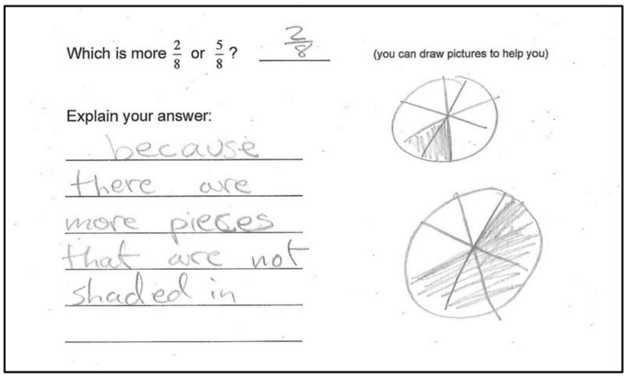


Figure 17

Maddie's written responses on the screener for the comparison problem of 2/8 and 5/8, in which she determined 2/8 was larger



Maddie

Unlike Ryan and Lily, there were no instances of Maddie demonstrating a halving understanding on either the screener or the interview. She did however demonstrate a fractional complement understanding on both. When asked to determine which quantity was more, she struggled particularly when the denominators of the fractions were the same. For example, she judged 2/8 to be larger than 5/8. Her solution helps illustrate how a fractional complement understanding was evident in this problem and how it was problematic (see Figure 17). Maddie drew canonical representations for both 2/8 and 5/8, using shading to represent the fractional quantity. However,

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she then judged 2/8 to be larger because there were "more pieces not shaded in." This highlights the disconnection between her canonical use of shading in her construction of the area models and her unconventional focus on the unshaded parts in interpreting her own drawings. The quantities she compared were not the quantities she herself drew, but the fractional complements.

Maddie again attended to the unshaded pieces when asked to compare area models of 4/5 and 3/5 (see Figure 18), incorrectly judging that 3/5 was larger because there were more parts not colored in. For both same denominator comparison problems, she incorrectly believed the smaller amount was larger, and in each case, she justified her answer by identifying that there was more that was not shaded.

In addition to these comparison problems, Maddie's fractional complement understanding was also evident when she interpreted the eight 1/10 pieces as 2/8 (pieces missing/pieces shown; see Figure 19).

Although it was not as evident during the interview, Maddie continued to rely on a fractional complement understanding. When asked to interpret a drawn area model of 4/5 (see Figure 20a), she, like Lily, interpreted it first in terms of the unshaded amount (1/4; unshaded/shaded). When asked to justify her answer of 1/4, she justified it by noting the number of boxes colored in, but did not change her answer. When the interviewer repartitioned this area model to produce 8/10 (see Figure 20b), she again initially focused on the two unshaded pieces. Unlike her previous answer, she eventually corrected this error. Throughout her explanations she vacillated between different interpretations of the representation. First providing a fractional complement answer (1/4) and justifying her answer with the shaded region, and then correcting her final interpretation (8/10) and justifying it based on the fractional complement.

Figure 18

Maddie's written responses on the screener on a comparison problem of 4/5 and 3/5 in which she was asked to circle the larger amount. She explains that 3/5 is larger because "there are two lines that are not colered in."

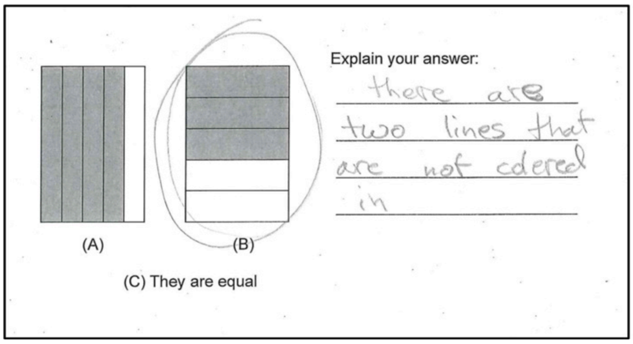


Figure 19

Maddie's written work interpreting eight 1/10 pieces in terms of the number of pieces missing (2) over the number of pieces shown (8)

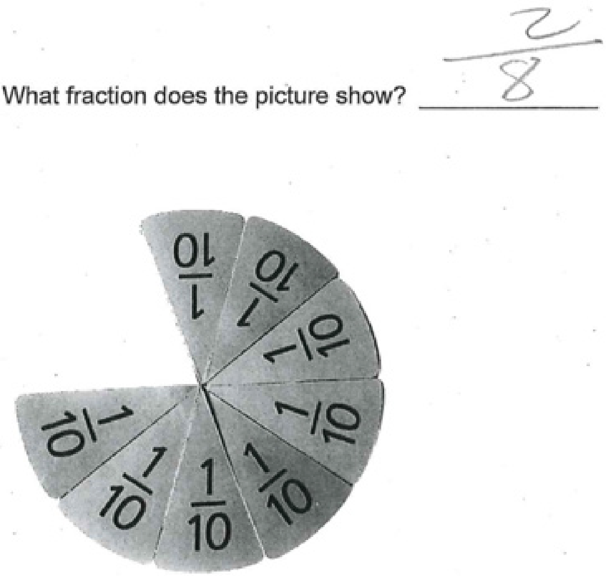
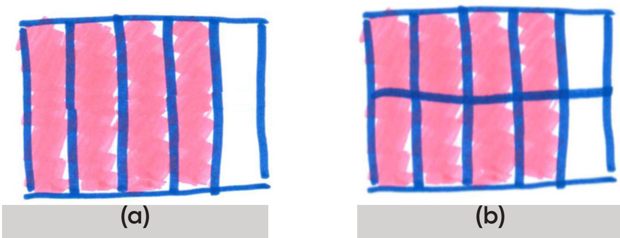


Figure 20

Tutor drawn representation of 4/5 (digitally recreated), which was then repartitioned to create 8/10



Interviewer: Okay, one student I was working with drew a picture like this. [draws rectangle with 5 sections, colors in 4; see Figure 20a] What would you say that's a picture of?

Maddie: I think that would be one-fourth.

Interviewer: How do you know?

Maddie: Because um, 4 – 4, I mean, um, 4 out of 5 boxes were colored in.

Interviewer: Okay, 4 out of 5 boxes were colored in. So then another student came along and cut it in half like that. [draws line down the middle; see Figure 20b]

Maddie: Um, that would be...

Interviewer: What would you say that is now?

Maddie: It would be 8 out of 2 – or, 2 out of 8. No, 4 out of 8. Wait. 8 out of 10. 8 out of 10.

Interviewer: 8 out of 10? How do you know?

Maddie: Because um, now that the squares are cut up, [touches picture], there are 8 that are colored and 2 that are left.

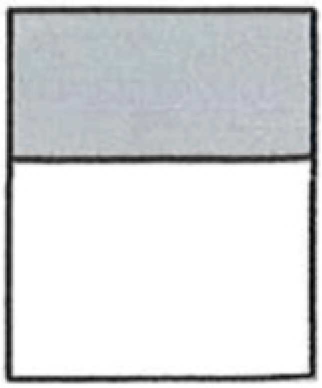
In her interpretation of 8/10, Maddie corrected her initial fractional complement answers (8/2 shaded/unshaded and 2/8 unshaded/shaded) and correctly determined that the repartitioned fraction was a representation of 8/10. However, she still attended to the fractional complement (2 pieces) and referred to them as "left." – one of the defining characteristics of the fractional complement understanding.

Maddie's focus on the unshaded space as the fractional quantity was also evident when asked to justify why she (correctly) did not select the unequally partitioned area model as a valid representation of ½ (see Figure 21). When asked why she did not select it, she interpreted the white (unshaded) part as the focal fractional quantity, and judged that the area model was more than ½.

> *Interviewer: Can you explain why you didn't choose this one?*
>
> *Maddie: Because the white has more of – the white is covering more of the square.*
>
> *Interviewer: So is this going to be less than one-half or more than one-half?*
>
> *Maddie: Um... [pause] I think it would be... [pause] I think it would be more. Um, because the white has more.*

Figure 21

Printed image that Maddie determined was more than 1/2



In her justification, Maddie understood this representation to be more than ½, suggesting that she was attending to the white space as the focal fractional quantity.

As in Ryan's case, there was some variation on the specific problems, which elicited her fractional complement understanding. On the screener it was primarily on comparison problems, and in the interview, it was primarily during interpretation problems. These data suggest that Maddie relied upon a fractional complement understanding to make sense of various fraction representations in various contexts.

Summary

All three students demonstrated unconventional understandings during the interview that were consistent with those documented in adults with dyscalculia (Lewis, 2014). Although there were often differences in the specific problems in which the understandings emerged, there was consistency in the nature of the understandings themselves. Maddie relied on a fractional complement understanding, and did so on both the screener and interview. Ryan and Lily demonstrated both a fractional complement and halving understanding. In Lily's case there was consistency in the problems and specific reasoning on the screener and interview, whereas in Ryan's case the same understanding persisted but with different frequencies and on different problems. We judge the screener to be a useful tool to identify students with these characteristic unconventional understandings given their high unconventionality scores on both the screener and interview. We then evaluated whether these three students met the standard criteria for dyscalculia classification established by the DSM-5.

Dyscalculia Classification

The DSM-5 requires that students with dyscalculia have persistent difficulties in mathematics that are evident during formal schooling and result in below average achievement. The DSM-5 recommends operationalizing "below average" as 1.5 standard deviations below the population mean on a norm referenced achievement test, which corresponds to the 7th percentile. Additionally, the student's low achievement must not be due to lack of educational opportunity, poor instruction, lack of fluency in instructional language, developmental delay, or a sensory, motor, or neurological disorder.

In order to evaluate whether these students also met the DSM-5 criteria for dyscalculia classification we considered students' composite and subtest scores on the Woodcock-Johnson Test of Achievement IV (WJ-IV) and self-reports of their educational history and opportunity. The WJ-IV scores for each student are presented in Table 3. Lily and Ryan clearly met the "below average achievement" criterion, as all of their subtests and composite scores were below the 7th percentile. Maddie's percentile scores were more variable. Maddie met the below average achievement criterion on only one subtest – Math Facts Fluency – and in one composite score (Mathematics Calculation Skills). Math Facts Fluency is the only timed math assessment within the WJ-IV, and researchers have argued for the importance of timed assessments of mathematics performance to accurately identify students with dyscalculia (e.g., Berch, 2005; Mazzocco, 2009). Indeed, when completing the untimed sections, Maddie's progress through the questions was laborious and time intensive. This suggests that she may have developed ways of compensating for her difficulties (see Lewis & Lynn, 2018 for a discussion), but that her difficulties were more evident under time constraints. Because Maddie's score on a timed assessment fell below the 7th percentile, we argue that she meets the dyscalculia criteria based on this more sensitive measure.

Table 3

Percentile scores on the Woodcock Johnson IV Test of Achievement for the case study students.

	Ryan	Lily	Maddie
Mathematics Composite	1	0.2	24
Broad Mathematics	<0.1	<0.1	8
Math Calculation Skills	<0.1	<0.1	7
Applied Problems	7	2	29
Calculation	1	0.1	25
Math Facts Fluency	<0.1	0.2	2

In addition to the below average achievement criterion, the students' self-reports indicate that these difficulties were evident in early school years, and the difficulties were not attributable to a global developmental delay, hearing, vision, neurological, or motor disorder. All students were White native English speakers (see Table 4) and therefore entered the school context with considerable privilege. Based on the individual self-reports all students had sufficient familial and educational resources (e.g., homework club, individual teacher/parent help), decreasing the likelihood that environmental or social circumstances were the origin of their difficulties in mathematics. These students were attending a private school for students with language-based learning disabilities, and although it is possible that their difficulties with language impacted their ability to learn mathematics, none of the students identified reading difficulties as an issue for them in mathematics.

Table 4

Demographic information for case study students.

	Ryan	Lily	Maddie
Gender	Male	Female	Female
Race	White	White	White
Age (years-months)	13-11	13-2	13-9
Grade	8	8	8

Conclusion

All three students who demonstrated high levels of unconventionality on the Screener continued to demonstrate these same unconventional understandings on the interview. This suggests that these understandings do persist over time and continue to lead to specific kinds of answers. All three students also met the qualifications for the DSM-5 dyscalculia criteria. This suggests that it may be possible to screen for characteristics of dyscalculia with a group administered screener.

Discussion

These two studies together provide a proof-of-concept for a novel approach to addressing the intractable identification issues facing dyscalculia researchers. Through these studies we provided a model for leveraging case study work in powerful ways to go beyond the individual cases and consider the prevalence of these patterns of understanding more broadly. By using detailed qualitative studies of extreme cases to design group administered written assessments, it may be possible to make considerable progress towards delineating the unique characteristics of this disability. This kind of approach is novel in that it attempts to define and identify dyscalculia by the unique characteristics (i.e., unconventional understandings) rather than defining dyscalculia as performance deficits.

Study 1 demonstrated that the unconventional understandings documented in Lewis (2014) were atypical. Only 6% of middle school students had high unconventionality scores. The percentage of students with high unconventionality scores was approximately equal to the estimated prevalence of dyscalculia in the general population (Shalev, 2007). The fact that (a) not all low achieving students demonstrated unconventionalities, and (b) that the students with the highest levels of unconventionality were not necessarily the lowest achieving students, suggests that the Screener identified qualitative differences in understanding, rather than simply low achievement.

Study 2 helped establish the validity of the Screener for identifying unconventional understandings. The students with high unconventionality scores on the Screener in study 2, did rely upon and demonstrate unconventional understandings in their interviews. Furthermore, additional assessments found that all three of these students met rigorous dyscalculia criteria established by the DSM-5. These studies together provide evidence that it may be possible to build off characteristic understandings documented in adults with dyscalculia to develop novel approaches for identification. Unlike standard approaches which struggle to differentiate dyscalculia from low achievement, these studies suggest that it may be possible to identify the characteristics of dyscalculia on a group-administered assessment.

Evaluation of the Screener

The validity of this Screening assessment was also evaluated through item factor analysis, which confirmed that this assessment measured two factors: halving and fractional complement. Although there was variability in how strongly particular items loaded onto the associated factor, we find analytic utility in all items. For example, although items 3 and 4 (draw 3/5; draw 1 5/8) did not load onto fractional complement, these questions did provide essential information for how the student understood the shading when drawing area models. If a student used shading to represent the numerator (i.e., fractional quantity) in their drawings, but used the unshaded parts to interpret the fractional quantity, it suggests an unconventional understanding of the shading. It is precisely because of the disconnection between how students draw and interpret area models that these items would not load strongly onto fractional complement, but nevertheless provide important information about the students' understanding. Similarly, although item 5 (interpret ½) did not load as strongly onto factor 1 (halving) we believe that this item provides important insight. For example, it was only on this item on the Screener that Ryan's tendency to understand ½ as halving was evident. The interview demonstrated that Ryan did rely upon a halving understanding when he drew non-shaded halves and identified the partition line itself as a representation of ½. Therefore, although some items did not load strongly onto the two factors, we believe they provide important insight into the students' understanding.

Future Research

We acknowledge that this Screener only includes a small subset of ways in which students with dyscalculia may understand mathematics in different ways. It is possible that additional research into how these students represent these fraction quantities on the number line (Schneider & Siegler, 2010) or compare fraction magnitudes (Meert et al., 2009) would yield insight into their understanding of fraction quantity. The field needs to invest in more detailed studies of extreme cases to specifically identify the characteristics of this disability across a range of mathematics topics. This suggests a dramatic shift from a focus on identifying performance deficits in speed and accuracy, to a focus on identifying what students with dyscalculia are doing and how these understandings may be unconventional. Until then, leveraging these characteristics may enable the development of alternative identification approaches. For example, if dyscalculia impacts students' learning across all mathematics topics (e.g., Lewis & Lynn, 2018) it may be possible to selectively recruit students with unconventional fraction understandings and then explore how these students make sense of other topics, like algebra.

Implications for Research and Practice

The issue of accurate dyscalculia identification has far reaching consequences for research and practice. Current use of the low achievement criteria has resulted in heterogeneous groups of students erroneously labeled as dyscalculic. Studies of dyscalculia that rely on this problematic and imprecise proxy are often studying low mathematics achievement – often due to inequitable educational opportunities – in the name of dyscalculia. The unintended consequences of this widespread use of this insufficient operational definition has resulted in myriad studies arguing that students with dyscalculia simply lag behind their peers (e.g., Gonzalez & Espinel, 2002; Keeler & Swanson, 2001; Mabbott & Bisanz, 2008). Because low achievement is used as the sole criteria for dyscalculia classification, studies have argued that students with dyscalculia are simply delayed in their mathematical development, rather than qualitatively different (Geary & Hoard, 2005). The developmental lag theory suggests the same teaching methods should be effective and these students simply require additional time and exposure to standard instruction. Because this research is largely based on studies which have not employed a sufficient exclusionary definition to determine that the low achievement is due to a disability rather than social or environmental factors (Lewis & Fisher, 2016), we take issue with this theory and its resulting implications for instruction.

In our studies we contribute to the growing body of work that suggests that qualitative differences in performance may be a productive approach to differentiate students with dyscalculia from students with low achievement due to other factors (e.g., Desoete & Roeyers, 2005; Mazzocco et al., 2008; 2013; Mazzocco & Devlin 2008). This suggests that a "more of the same" instructional approach will not work for these learners, because they have difficulties that are qualitatively different than their peers. We argue that the unconventional understandings identified in the Screener and Interview impact a student's ability to access standard instruction and these students may require different kinds of instruction that takes these issues of access into account (Lewis, 2017). At the heart of both unconventional understandings is a tendency to understand representations of quantities as representations of action (e.g., taking or halving). Students who rely upon these kinds of qualitatively different unconventional understandings require alternative forms of instruction that acknowledge and build upon these students' unique resources (Lewis, 2017).

If used in practice, this Screener should just be used as a first step in a holistic evaluation of the student. All students may experience unconventional understandings when first learning how to use and translate between different mathematical representations (symbols, language, and pictorial; Viseu et al., 2021), so this Screener may not be effective with younger students first learning about fractions. For students with adequate opportunity to learn about fractions, persistent evidence of unconventional understandings may signify an issue of access. For students with suspected dyscalculia, multiple

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assessments including observation, interview, and other nonstandard assessment are recommended to determine if the difficulties are due to dyscalculia or other factors (Mundia, 2017). These kinds of nonstandardized assessments help educators identify unconventional understandings, issues of access, and suggest how to design alternative accessible instruction for that student (e.g., Lewis, 2017).

Limitations

There are several limitations of the current study. First, this assessment was limited to exploring basic representation and interpretation of fraction quantities, which represent a narrow slice of fraction concepts and skills. Although some researchers might argue that the narrow topic domain is problematic because mathematics is componential in nature (Dowker, 2015), we argue that these unconventional fraction understandings are indicative of underlying number processing issues, representing quantities as actions, rather than objects (Sfard, 1991). We do not claim that the Screener captures the myriad ways in which dyscalculia may manifest, however, students who have been identified using this screener have had similar unconventional understandings when working with integers (Lewis et al., 2020) and algebra (Lewis et al., 2022), suggesting the utility of identifying these kinds of unconventional understandings even in a narrow topic domain. We do not propose the Screener to be a test for dyscalculia, instead these studies are intended to illustrate the potential utility of a general approach to drawing upon evidence of unconventional understandings identified in detailed analyses of extreme cases to design more sensitive screening tools.

A second limitation of this study is that in study 2 the dyscalculia criteria were assessed only for students who were attending a school for students with language-based learning disabilities. It is possible that the students' language-based learning disability did impact their understanding of mathematics. There is specific academic language associated with fractions (e.g., numerator, denominator; Bossé et al., 2019), and it is possible this created an additional barrier for students. We cannot fully address issues of comorbidity that this participant population raises. However, in other preliminary work, there is some evidence that the Screener works to identify college-aged students with dyscalculia with no other learning disabilities (Lewis et al., 2020). Future work should consider whether this kind of screener has utility for identifying students without other learning disabilities in a general population of students.

Third, although we documented unconventional understandings in the case study students, it is an open question what kind of instruction would be necessary to support their understanding of fractions as quantities. Although research has demonstrated this kind of re-mediation with one of the adult students from the first case study (Lewis, 2017), more research is needed to determine if similar approaches would be effective for younger students.

One final limitation, is that due to the nature of the anonymous data collection for Study 1, we relied upon the teachers recording of test scores on the written assessments. These are the only data that were not double coded, and therefore, inadvertent errors could have been made. Because this was an ancillary point and not the main objective of the study, this potential for error in the data was not seen to be critical.

Conclusion

These studies established a proof-of-concept for designing a group administered screener by leveraging the qualitative differences identified in students with dyscalculia. This provides a novel approach to address the long-standing methodological issues facing the field with regards to identification and classification of students with dyscalculia. We believe that conceptualizing dyscalculia in terms of developmental difference rather than deficit has the potential to greatly impact both research and practice for students with dyscalculia. The screener identified students who understood standard tools for representing fractions (drawings, symbols) in ways that were unconventional and would render these standard mediational tools inaccessible. This suggests that instruction which relies on these standard representations would be inaccessible and that alternative more accessible instruction may need to be designed. Students who score high on this screener are worthy of further assessment to evaluate how to support their fractions learning and to determine if they have other issues of access across other topic domains.

[1]The terms "dyscalculia" and "mathematics learning disability" are used interchangeably in the field (Mazzocco, 2007). We use the former because this term is more commonly used internationally. We differentiate dyscalculia – which involves a difference in how the student processes numerical information – from students with mathematics learning difficulties who may have low achievement in mathematics due to a variety of social or environmental causes.

[2]Response-to-intervention approaches, which are sometimes used in schools to identify students who qualify for special education services, are not often used in research on dyscalculia because they lack specificity and methodological rigor. A small number of studies (2%, based on a systematic literature review; Lewis & Fisher, 2016) have used growth curve analysis

to identify students who not only are low achieving, but also have slow growth, however this kind of Response-to-Intervention approach is not commonly used in the field.

[3]It is worth noting that this pathologizing of human variation can be thought of as problematic, and this delineation of humans into "normal" and "abnormal" has its origins in the eugenics movement (e.g., Davis, 2006). The point here is not to take a position on whether the category of dyscalculia is morally, ethically, practically, or politically appropriate, but to identify that when disability categories have been defined, it has often started with the close and careful clinical appraisal of individuals considered to be exceptional. In this study our goal is not to further pathologize human variation, but to better understand how cognitive differences may result in inaccessibility in mathematics. By improving identification approaches we hope to (a) enable students with this disability to advocate and obtain access to accommodations to address the inaccessible mathematics context and (b) avoid inappropriately labeling students with low mathematics achievement as disabled.

[4]Although the fractional complement for 3/4 is 1/4, we also classified instances where the student interpreted the fraction as unshaded/shaded (e.g., 1/3), because their answer suggested that the student was attending to the fractional complement (the one unshaded part) as the focal quantity.

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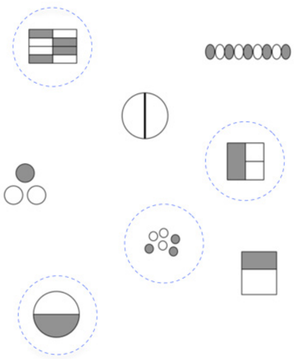
Appendix A

Questions and scoring criteria for the Screener

#	Question	Correctness	Unconventional Points	Not Unconventional
1	Draw a picture of $\frac{1}{2}$	1 point for a canonical drawing or representation of ½ (i.e., area model, number line, decimal, percent, or semicircle)	1 unconventional point for a drawing of a shape partitioned into two parts but without shading or labeling of either part.	A drawing of a shape partitioned into two with one of the parts labeled "½"
2	Draw another way to show $\frac{1}{2}$	(same as above)	(same as above)	(same as above)
3	Draw a picture of $\frac{3}{5}$	1 point for a canonical representation of 3/5 (e.g., area model, number line, or discrete set).	1 unconventional point for an area model or discrete set drawing of 2/5 (i.e., 3 out of 5 parts unshaded).	Partitioning issues, because students have difficulty accurately partitioning into fifths.
4	Draw a picture of $1\frac{5}{8}$	1 point for a canonical representation of 1 5/8 (e.g., area model, number line, or discrete set).	1 unconventional point for a representation where the whole is not shaded or labeled.	A drawing where the wholes are different sizes.
5	Circle all the pictures that you think show $\frac{1}{2}$? (correct answers circled below)  (adapted from Ni, 2001)	1 point for each correctly circled canonical representation of 1/2 (see circled answers) -1 point for each incorrect answer.	1 unconventional point for circling the halved circle with no shading.	
6	Which is more $\frac{1}{6}$ or $\frac{1}{8}$? (you can draw pictures to help you) Explain your answer:	1 point for correct answer (1/6) (explanations are used to disambiguate student answer, not required)	1 unconventional point for incorrect answer (1/8) with an explanation and/or drawing that focuses on the unshaded amount (e.g., more left in the 1/8 drawing).	A written answer that states that 1/8 is bigger than 1/6 because 8 is bigger than 6.
7	Which is more $\frac{2}{8}$ or $\frac{5}{8}$? Explain your answer:	1 point for correct answer (5/8) (explanations are used to disambiguate student answer, not required)	1 unconventional point for an incorrect answer (2/8) with an explanation and/or drawing that focuses on the fractional complement (e.g., 6/8 unshaded, 3/8 unshaded).	Answer of 2/8 with no explanation or drawing.

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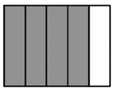
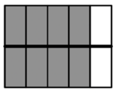
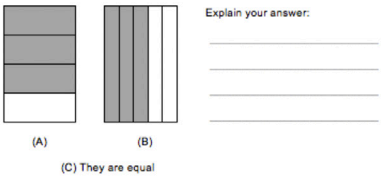
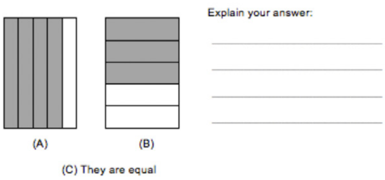
8	Lisa drew this picture.  What fraction does this drawing show? ______	1 point for correct answer (4/5).	1 unconventional point for answers where the numerator is the number of parts not shaded (e.g., 1/5 or 1/4).	Answers where student has miscounted the number of pieces, (e.g., 5/6)
9	She then cut it like this  What fraction does this drawing show now? ______	1 point for correct answer (8/10 or 4/5).	1 unconventional point for answers where the numerator is the number of parts not shaded (e.g., 2/10, 2/8, 1/5 or 1/4).	Answers where student has miscounted the number of pieces, (e.g., 10/12).
10	Which is bigger? (circle your answer)  (adapted from Armstrong & Larson, 1995)	1 point for correct answer (A) (explanations are used to disambiguate student answer, not required)	1 unconventional point for selecting B with an explanation focusing on the number "left" or unshaded amount.	An incorrect answer (B or C) with either no explanation or an explanation that suggests miscounting, (e.g., "C because 3/5=3/5")
11	 (adapted from Armstrong & Larson, 1995)	1 point for correct answer (A) (explanations are used to disambiguate student answer, not required)	1 unconventional point for selecting B with an explanation focusing on the number "left" or unshaded amount.	An incorrect answer (B or C) with either no explanation or an explanation that suggests miscounting, (e.g., "C because 3/5=3/5")
12	Solve the problem $\frac{1}{2} + \frac{1}{4} =$ using pictures.	1 point for correct answer (3/4). Student not required to draw pictures.	1 unconventional point for (a) answers that include a drawing of ½ without shading or (b) an answer of 2/4 (unshaded/shaded) with canonical area models of ½ and ¼.	An incorrect answer of 1/6 or 2/6 are not considered unconventional by themselves.
13	What fraction does this picture show? 	1 point for correct answer (e.g., 8/10 or 4/5).	1 unconventional point for an answers that determine the numerator based on the missing pieces (e.g., 2/10, 2/8, 1/5, 1/10, 1/8).	An incorrect answer in which the student has miscounted (e.g., 7/10 or 9/10).
Global coding: Any time the student interpreted a representation of as the fractional complement (e.g., interpreting 2/3 as 1/3) the student got an unconventional point for that problem.				

Appendix B

Background Interview Questions

Academic Background:

- What is your favorite subject in school?
- What do you like about it?
- What is your least favorite subject?
- What don't you like about it?
- What do you think about math? What do you like about it? What don't you like about it?

Nature of the student's difficulty:

- What are you working on in math right now?
- Can you give me an example?
- What about learning and/or doing math was hard for you? Can you give an example?
- What about learning and/or doing math was easy for you? Can you give an example?

Effort:

- Do you get a lot of homework in math?
- When do you do your homework?
- Do you tend to do all your homework and turn it in?

Resources Questions:

- If you get stuck on a problem, what do you do?
- Who do you ask for help, if you need it?

Language Fluency:

- What language do you tend to speak at home?

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Elementary Students' Exploration of the Structure of a Word Problem Using Representations

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Abstract

Word problems are frequently used in school mathematics to offer students the opportunity to explore mathematical relationships and structure. However, previous studies have reported that word problems are misused or abused in ways that overlook the original intent of exploring mathematical structure and relationship. This study aims to share a series of a small group of third-grade students' explorations while debating the mathematical relationships in solving a word problem with representations over several days. Although the exploration took longer than planned, it was worthwhile. It offered students a space to express confusion, showcase their knowledge, test conjectures, and imagine alternative contexts. Ultimately, these explorations helped students recognize multiple relationships within the context of specific problems while bringing their attention to real-world related applications. The retrospective analysis of class episodes offers insight into learning opportunities to support students in exploring mathematical structure and relationships while discussing and debating the word problem context.

Keywords:

Additive and Multiplicative Relationships, Classroom Culture, Elementary Education, Schematic Representation, Structure, Word Problems

Introduction

Understanding and generalizing mathematical relationships and structures in learning mathematics are critical (Davydov, 1990; Mason, 2003; Sierpinska, 1994; Thompson, 2011). The ability to "look closely to discern a pattern or structure" is an essential skill that mathematics learners should develop (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 8). Also, "the detection and exploitation of structural relationships" is considered an essential mathematics component (Greer & Harel, 1998, p. 22).

Word problems are frequently used in school mathematics to offer students the opportunity to explore mathematical relationships and structure. However, studies have reported that students face varying challenges and difficulties in handling world problems (Verschaffel et al., 2020). Researchers discussed situations in which word problems are misused or abused, resulting in blocking the intended results of mathematical exploration (e.g., Mason, 2001; Verschaffel et al., 2000). In particular, studies reported that many students tend to dive into calculations by grabbing given numbers and using known procedures and operations or rely on keywords, rather than analyzing the structure of the problem as a means to solve the problem (Littlefield & Reiser, 1993; Savard & Polotskaia, 2017; Stigler et al., 1990; Verschaffel et al., 2000). Additionally, students generally produce one answer in the form of a numerical symbol and seem unwilling to bring anything further into the problem-solving process. Students also believe that there is only one correct answer or one correct process for finding a solution for a word problem. These two tendencies often keep students from paying attention to the context of the word problem, while these students generally have difficulty with problem-solving (Schwieger, 1999).

Educators employ various approaches to help students pay attention to and analyze the mathematical structure of a problem. For example, students are often encouraged to represent or model the relationships in ways that allow them to manipulate the quantities and reveal the structure, supporting their discovery of the required arithmetic operation. Some researchers supported schematic-based instruction, claiming that schematic diagrams better serve students (e.g., Terwel et al., 2009). Several studies highlighted a conceptual correlation between schemas and problem-solving (e.g., Jitendra & Star, 2011; Steele & Johanning, 2004; van Garderen et al., 2013). Other researchers offered activities that help students discern different word problem story grammar (e.g., Xin, 2012).

Previously, mathematics educators focused on teaching predefined schematic representations based on cognitive psychology (Fagnant & Vlassis, 2013). They asked students to memorize several predefined representations to solve certain types of problems, and students were expected to develop an ability to categorize problems based on the representations used (Schoenfeld, 1992). However, with increased attention to sociocultural perspective (Cobb & Hodge, 2011) and mathematical process (National Council of Teachers of Mathematics [NCTM], 2014), today's mathematics educators are encouraged to provide students with opportunities to examine mathematical structures of problems and represent them through student-oriented investigations (Fagnant & Vlassis, 2013). For example, NCTM (2014) suggests to "allow students to select and discuss their choices to represent the problem situations" (p. 28). When students gain authority in their mathematics investigation, they can make sense of connections between representations, understanding central mathematical ideas, and experiencing authentic mathematical problem-solving processes. Teachers should encourage students to engage in mathematical discussions about using and understanding schematic representation to improve students' problem-solving abilities in word problems.

In short, despite the many possible supporting tools and approaches, "what seems to matter most is not the apparatus itself, but how it is used" (Mason, 2018, p.332). Good tools and approaches to real-world problems, such as the aforementioned ones, can be (and often are) incorrectly presented through a teacher-led, top-down presentation rather than as an apparatus for student-centered exploration. When students have more opportunities to play with, be curious about, and explore word problems by changing the context and numerical parameters, it can be more enjoyable for them to explore structural relationships in the context of a word problem (Mason, 2018).

This study shares a series of explorations undertaken by a small group of third-grade students over several days. Using schematic representations and real-world examples, this group discussed the mathematical relationships involved in the following story problem: "A father is 32 years old, and his son is 4 times younger than him. How old will they be in 4 years?" The purpose of this study is to show the students' exploration process through three vignettes while providing interpretational space for readers. In this retrospective analysis of teaching episodes, this study focuses on the following questions: (a) What types of confusion and curiosity did the students exhibit while using schematic representations to identify the additive and multiplicative relationships? (b) How did the students make sense of the mathematical relationships underlying schematic representations? (c) What kind of classroom culture should be established to support student reasoning and justification?

Literature Review

Problem Structure

Although the term "structure" has been widely used in mathematics education without clear definitions, researchers consider knowledge about structure as an awareness of a network of local and general relationships (Venkat et al., 2019). Venkat et al. (2019) noted that emerging structures involving analyzing, forming, and seeing local relationships can be observed when young students analyze and distinguish local relationships, ultimately allowing them to identify mathematical structures with more general mathematical relationships and properties. For young mathematics students, exploring the different potential structures embedded in additive and multiplicative situations is a critical pathway for developing students' understanding and ability to operate within these structures flexibly (Mason, 2018).

Typically, curriculum using word problems includes multiple structures within additive and multiplicative situations. It is common that students make additive errors in multiplicative missing-value word problems and multiplicative errors in additive missing-value word problems.

Researchers highlighted that an important difference between additive and multiplicative relationships is the nature of invariance (Behr & Harel, 1990; Degrande et al., 2019). In other words, quantities are linked additively in the additive structure, and the actual difference between quantities remains invariant. In contrast, the ratio (e.g., relative difference) of quantities linked multiplicatively (e.g., linked through multiplication and division), what is invariant is the ratio between quantities.

> *When considering the word problem at hand, we can consider several structures.*
>
> *A father is 32 years old, and his son is 4 times younger than him. How old will they be in 4 years?*

First, the given relation ("4 times younger") supports students in multiplicative reasoning. Thus, the son's current age is 8 because 32 divided by 4 is 8. Second, the question turns students to additive reasoning. After finding the son's age, students can find missing values ("in 4 years") through different strategies. As the examples below show, 4 years are added to the current ages of father and son:

> *Father's age in 4 years: 32 + 4 = 36*
>
> *Son's age in 4 years: 8 + 4 = 12*

Alternatively, noting the actual difference between the current ages of father and son, 24, the final missing value is identified as follows:

> *The difference between the current ages of father and son: 32 – 8 = 24*
>
> *Father's age in 4 years: 32 + 4 = 36*
>
> *Son's age in 4 years: 36 – 24 = 12*

Thus, the situation can be explained differently depending on the relationships students recognize.

Representing the Problem Structure

Researchers noted that students have difficulties in understanding structures and analyzing quantitative relationships of word problems (Mason, 2018). Several researchers highlighted the importance of visualizing and representing the problem contexts to support students' attention and analyze the structure and relationships underlying a problem. Therefore, in mathematics curricula and programs, it is prominent to include various representations of a problem to elicit the structure and relationships within it. For example, materials used in the Math Recovery Program (e.g., Wright et al., 2006) frequently incorporate five or 10 frames to help students' early numeracy knowledge by illustrating the structure of numbers and the place value concept. In the Singapore curriculum, various models, such as bar models, support students' deeper understanding in solving word problems (Kaur, 2019; Ng & Lee, 2005, 2009).

More explicit use of models in elementary mathematics curriculum can be found in Davydov's curriculum (Davydov et al., 1999), where the critical role of symbols and models is emphasized. In the latter curriculum, students manipulate real objects and graphic models such as line segments and schematics to represent implicit and explicit structural relationships. As they progress, the use of concrete objects and graphic models decreases, and the use of symbolic formulas increases. For instance, physical objects or graphic models of a part-whole relationship help students initially see all involved quantities and their connection. Later, students can formulate algebraic equations for this mathematical relationship (Lee, 2002; Schmittau, 2005). Several studies reported the effectiveness of using various tools to represent and visualize relationships between quantities when solving word problems (Kaur, 2019; Ng & Lee, 2005, 2009; Schmittau, 2005).

Word Problem and Schematic Representation

Mathematics educators highlighted the importance of word problems in learning mathematics (NCTM, 2000; van Garderen et al., 2013; Vula et al., 2017). Word problems refer to problems that are "typically composed of a mathematics structure embedded in a more or less realistic context" (Depaepe et al., 2010, p. 154). Word problems help students construct mathematical representations and understand mathematical relationships and structures. They help them explore the relationship between reality and abstract mathematical concepts and operations (Jitendra, 2019). Studies showed that students usually go through problem-representation and problem-solution phases to solve word problems (Depaepe et al., 2010; Jitendra, 2019). In the problem-representation phase, students comprehend the problem and construct representations (or models) to illustrate the problem situation clearly. However, students work through the constructed representations in the problem-solution phase and interpret and evaluate the outcome.

In a well-known classification scheme for representation types, Lesh et al. (1987) emphasized flexibility and variability in meaningful use of representations among contextual, visual, verbal, physical, and schematic (or symbolic) representations. The visual representation retains most of the detailed information of the original contexts and clearly

represents concrete visualization of objects to help students understand the problem contexts (Hegarty & Kozhevnikov, 1999; Viseu et al., 2021). However, schematic representations abstractly represent a structural relationship of mathematical elements in a problem. As schematic representations are "meaning-based representations" (Terwel et al., 2009, p. 27), they discard unimportant information and select mathematically important relationships and structures used in the problem-solving process. Therefore, students are expected to convert verbal information into symbolic expressions, such as line, diagram, and shapes, and use them to construct arithmetic operations during the problem representation phase.
Some studies reported that mathematics educators often introduced predefined schematic representations and asked students to memorize those representations to solve word problems (Fagnant & Vlassis, 2013). However, findings of some studies revealed that solving word problems with representations does not always increase students' performance (Diezmann & English, 2001; Terwel et al., 2009; Verschaffel et al., 2020). For example, Terwel et al. (2009) examined the effect of teacher-provided representations on solving word problems with fifth-grade students and reported minimal improvement in student problem-solving abilities. However, their counterpart group, which was asked to construct representations through collaboration, showed considerable improvement. As the reasons for these different outcomes, the researchers explained that the collaboration allowed students to improve their understanding of problem structures and enhance students' capabilities to generate new problem-solving strategies. Similarly, Lehrer et al. (2000) examined elementary school students and found that student-generated representations were more beneficial for developing their conceptual competence than using teacher-sanctioned representations. However, these findings did not reveal that teachers should not teach schematic representations to their students; instead, it means that teachers should first give students opportunities to learn and use predefined representations. Teachers should then allow students to construct their schematic representation based on their understanding paired with thoughtful discussion and analysis among classmates (Diezmann & English, 2001; Lehrer et al., 2000).

Previous studies have largely adopted quantitative research methods to examine the effect of employing schematic representations on students' word problem-solving abilities. Thus, limited qualitative information on what types of confusion and curiosity is exhibited by students when using schematic representations. We also lack understanding of how students make sense of mathematical relationships underlying schematic representations, and there is little guidance on what types or aspects of classroom culture should be established to best support students as they learn word problems. Therefore, further studies can be conducted to examine students' exploration of mathematical structures of word problems with representations.

Methods

Context and Participants

The class episodes were taken from a three-year teaching experiment conducted in a private school in the US (Lee, 2002). The first author taught a cohort of seven students using the first three years of elementary mathematics curriculum developed by Davydov and his colleagues (Davydov et al., 1999). There were two male students and five female students. For five students, this private school was their first formal education setting, and two students had some public school experience. There were three or four mathematics classes per week, and each class session lasted approximately 50–60 minutes. The curriculum consists of a series of problems. Students were accustomed to engaging in an in-depth discussion (or debate) on a small number of problems each session.

The class was in the third year of the experiment when discussing the word problem that this study discusses. Prior to this discussion, the students were accustomed to using literal variables, while they had the freedom to refer to known or unknown quantities using some tools such as question marks, blanks, underlines, or verbal descriptors. These students were also accustomed to problems that were impossible to solve due to insufficient or contradictory information. Such problems aimed to facilitate the students' justification and reasoning process. The students called them trap problems (Lee, 2007). The students were also familiar with using various representations such as line segments and schematic representations. Students used self-invented schematic representations at times, but they usually used mutually agreed-upon representations. Figure 1 shows some examples. As shown, students were encouraged to relate various relationships by analyzing the structure of the given schematic representations.

When the class episodes in the following section occurred, the students had already studied additive and multiplicative relationships and analyzed various contexts (word problems). In previous experiences, the problem contained only one relationship — either the additive or multiplicative relationship. Thus, the invariance of difference or ratio was maintained. The discussion presented in this study happened when students needed to consider both additive and multiplicative relationships in the same context.

Data Sources and Data Analysis

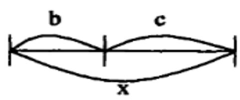
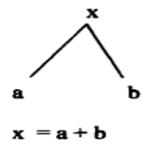
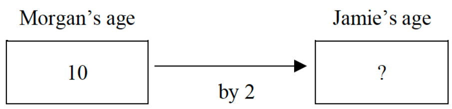
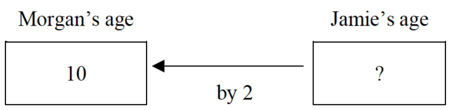
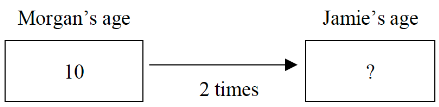
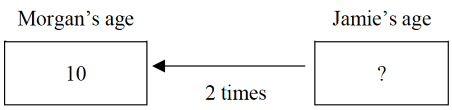
The primary data sources of the study were classroom discourse and field notes that the first author documented after each class session, describing interactions between the students, the teacher, and among students. For this study, the authors focused on the three days of class vignettes related to the discussion on the given word problem. A descriptive case study design (Yin, 2003) was used to examine student challenges during word-problem solving, and how students resolved those challenges through a series of small group discussions.

A case study examined a few cases of a phenomenon in a real context (Creswell & Poth, 2016). As individual cases are strongly connected in space and time, it is important to examine their context. A descriptive case study clearly describes a phenomenon and focuses on tracing "the sequence of interpersonal events over time (and describing) a subculture of it" (Yin, 2003, p. 4). For example, if researchers investigate the development of students' interactions over time, they can examine student participation and discourse, as well as their teachers' roles and discourse, to get a complete picture of the classroom environments.

The second author had an unbiased third-party role. As the first author was a teacher and thus directly participated in the classroom interactions, the second author also independently examined the raw data. Then, the two authors collaborated during several online meetings to compare and discuss the interpretation of the raw data at hand.

Figure 1

Examples of Schematic Representations Students Used

Schematic Representation	Example	Related symbolic representations
Line segment for a part-whole relationship		x = b + c b + c = x x - b = c x - c = b
Schematic for a part-whole relationship		x = a + b a + b = x x - a = b x - b = a
Schematic for the additive relationship	Morgan is 10 years old. Jamie is 2 years older than Morgan. How old is Jamie? 	10 + 2 = ? ? - 2 = 10 ? - 10 = 2
	Moran is 10 years old. Jamie is 2 years younger than Morgan. How old is Jamie? 	10 - 2 = ? 10 - ? = 2 ? + 2 = 10
Schematics for the multi-plicative relationship	Morgan is 10 years old. Jamie is 2 times older than Morgan. How old is Jamie? 	10 × 2 = ? ? ÷ 2 = 10 ? ÷10 = 2
	Morgan is 10 years old. Jamie is 2 times younger than Morgan. How old is Jamie? 	10 ÷ 2 = ? ? × 2 = 10 2 × ? = 10

This process helped ensure the credibility of this study and provided the two authors the opportunity to retrospectively analyze key learning instances that occurred during the process (Yackel, 2001) from insider and outsider perspectives. More specifically, we took the following steps: (a) independently review classroom discourse and descriptive field notes and identify what important elements and moments, (b) compare each other's elements and moments, (c) jointly identify the most salient themes, and (d) juxtapose the data with interpretations based on extant literature. In short, we allowed the analysis to emerge from our understanding and interpretations of the events unfolding on the data, instead of approaching the data with a predetermined coding scheme. The reliability of the analysis is not obtained by the coincidence of interpretations among us. Instead, we directly presented classroom discourse to increase the external validity and transparency of the study (Creswell & Poth, 2016).

Descriptions of Class Episodes

The following three vignettes taken from the first author's field notes show the students' confusion regarding the use of schematic representations, differentiation between additive and multiplicative relationships and the critical moments in which the students chronologically shifted their attention. The class episodes also show how students make sense of the mathematical relationships underlying schematic representations. In the teaching episodes, the teacher's role was minimal and focused only on facilitating the discussion and recording (or helping record) students' discussions in visual forms. All students' names are pseudonyms.

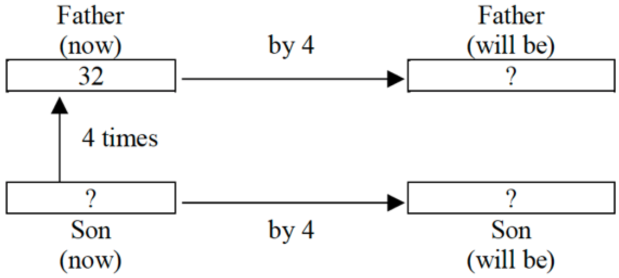
Vignette 1

"A father is 32 years old, and his son is 4 times younger than the father. How old will they be in 4 years?"

To solve this question, the student first determined the known (father's age now) and unknown quantities (son's age now, father's age in 4 years, and son's age in 4 years) and the relationships between given quantities. Based on that, the students drew the following schematic representation to help class discussion (see Figure 2). Students used descriptors for the quantities instead of literal variables in the schematic representations. In students' terms, the father's age in 4 years was noted as "father will be" and son's age in 4 years as "son will be." Unknown quantities were noted using question marks. Students noted the additive relationship between quantities "by" and multiplicative relationship "times."

Figure 2

Initial Schematic Representation



Using the information provided, students were able to find the values for all unknown quantities, as illustrated in Figure 2.

- [Father now]: 32 years old
- [Son now × 4]: 32, [Son now]: 32 ÷ 4 = 8
- [Father will be]: 32 + 4 = 36
- [Son will be]: 8 + 4 = 12

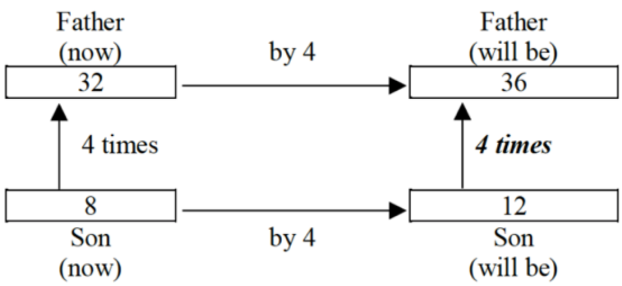
Meanwhile, the students found the values of all unknown quantities and completed the problem. However, one student, Jordan, started talking about the relationship between Father will be (father's age in 4 years) and Son will be (son's age in 4 years), noting that they did not show the relationship between these two quantities.

> Jordan: "Father (will be) is 4 times older than his son (will be)."
>
> Chris: "The problem did not ask for that relationship."
>
> Jordan: "However, we know that the relationship between the father's age and the son's age stays the same."

Although Chris resisted to do extra work (not due to any mathematical reason), students agreed that they could put "4 times" in the schematic representation between the father's age and son's age in 4 years (Figure 3).

Figure 3

Examining the Unasked Relationship



When completing the problem, students realized that something was wrong in the schematic representation because the father will be in 4 years (36) was not 4 times greater than the son will be in 4 years (12). At this point, Chris again suggested deleting "4 times" between the father's and son's ages in 4 years. Chris believed that we were not responsible for explaining the relationship between father will be and son will be by deleting the connection between them.

Other students disagreed with Chris, stated that the relationship would still exist even after it was deleted. Then, they concluded that this was a trap problem due to the contradictory information. Chris also agreed that it was a trap problem because the relationship between the father's age and the son's age should stay the same in 4 years. However, he continued to argue that there was no need to talk about this issue as the question did not ask about this relationship.

Vignette 2

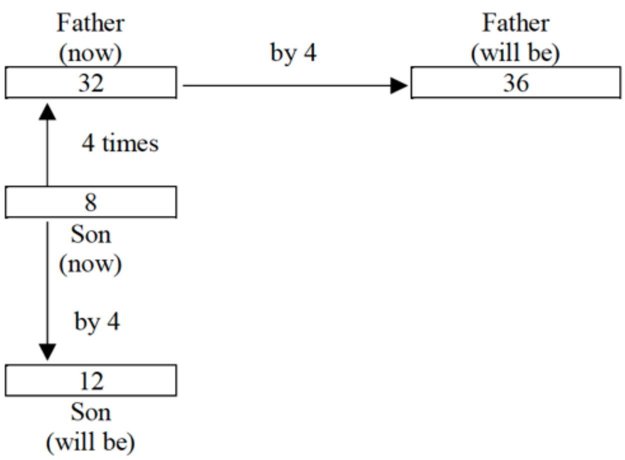
The discussion began with a review of previous day's conclusions as to why this problem was a trap:

The relationship between the father's age and the son's age should remain the same even after 4 years. The father is always 4 times older than the son. However, here, 36 divided by 12 is 3, not 4. It does not make sense. So, it is a trap.

While all students agreed on this conclusion made in the session, one student, Morgan, changed her mind. Morgan stated, "I changed my mind. I think it is not a trap." When asked to explain, Morgan redrew the previous schematic representation as an attempt to disconnect father (will be) from son (will be), as depicted in Figure 4.

Figure 4

Morgan's First Attempt to Change the Shape of the Schematic Representation

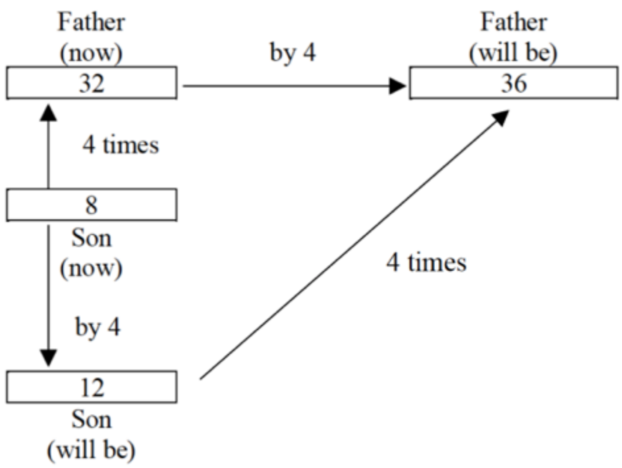


Morgan tried to justify her argument by changing the shape of the schematic representation, believing that father will be and son will be cannot be connected in this situation. This attempt was more likely to support Chris' argument in Vignette 1. While Chris argued that we did not need to find that relationship in the problem, Morgan tried to show that it was not possible to find the relationship by changing the shape of the schematic.

> *Morgan: "We cannot connect father (will be) and son (will be) now, so we cannot say that it is a trap."*
>
> *Chris: "But we can still figure out the relationship between father will be and son will be, and it should be the same as the relationship 4 years ago."*
>
> *Morgan: "We don't need it."*
>
> *Chris: "We don't need it, but we can do it. (Chris connected father will be and son will be and noted the relationship as "4 times" as illustrated in Figure 5).*

Figure 5

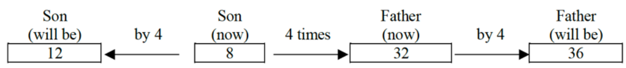
Peers' Reaction to Morgan's First Attempt



Morgan, then drew another schematic on the board, as illustrated in Figure 6.

Figure 6

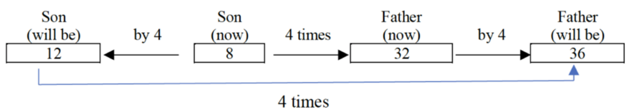
Morgan's Second Attempt to Change the Shape of the Schematic Representation



> *Morgan: "Now, we cannot connect father (will be) and son (will be)."*
>
> *Jamie: "Still, we can connect them, and we can write '4 times' there" (she added a line between father will be and son will be, as shown in Figure 7)*

Figure 7

Peers' Reaction to Morgan's Second Attempt



The students rejected Morgan's conjecture that the shape of schematic representation would make a difference. However, they could not find why the

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relationship between father will be and son will be were not 4 times bigger or smaller. Thus, this continued to be a trap problem.

Vignette 3

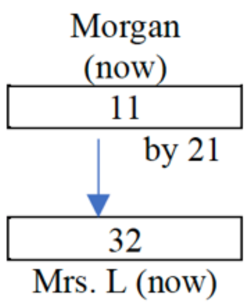
In this session, a student suggested taking a different example. Alex tried another example using Morgan's age and the age of an adult in the classroom. We drew a schematic on the board along with Alex's explanation.

> *Alex: "Morgan, how old are you?"*
>
> *Morgan: "11."*
>
> *Alex: "Mrs. L., how old are you?"*
>
> *Mrs. L.: "32."*

At this moment, the teacher asked the students how we could write this relationship in the schematic. The students answered easily (by 21), and it was recorded in the schematic (Figure 8).

Figure 8

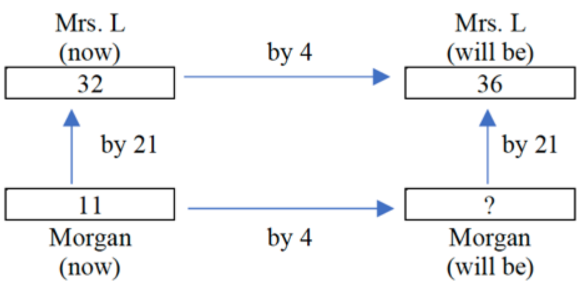
Alex's New Example



> *Alex: "Morgan, how old will you be in 4 years?"*
>
> *Morgan: "15."*
>
> *Alex: "How old will Mrs. L be in 4 years?"*
>
> *Morgan: "36."*
>
> *Alex: "OK. What is the age difference between you and Mrs. L in 4 years?"*
>
> *Morgan: "21."*
>
> *(Again, students wrote 'by 21' in the schematics.)*
>
> *Alex: "See, the age difference did not change." (Figure 9)*
>
> *Morgan: "Yes, but we don't need to figure that out."*

Figure 9

Alex's New Example: Expanded Version



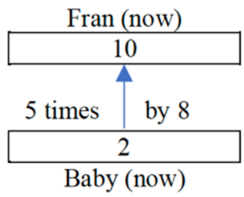
Although Alex's explanation was good and the students used the phrase "age difference," they were unable to connect the different explanations for this problem and the original problem regarding the ages of father and son in 4 years. At this point, the teacher encouraged the students to think about an additional problem in a similar context.

> *Teacher: "Alex's explanation was very interesting. Can we make another example? Can you use Fran's age and Mrs. L's baby's age this time?" (They all knew that Fran was 10 years old, and Mrs. L's daughter was two years old.) What is the relationship between Fran now and Mrs. L's daughter now?"*

Interestingly, this time, some students said, "by 8," and some of them said "5 times." The teacher wrote down both relations in the schematic (Figure 10).

Figure 10

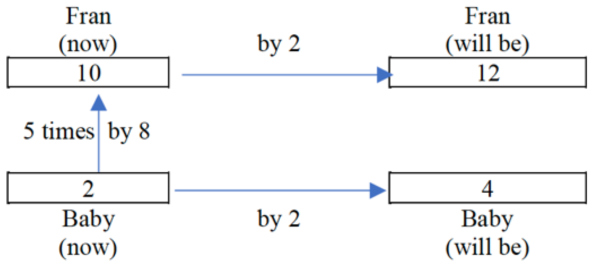
Teacher's Variation Problem



> *Teacher: "Let's think about their ages in two years."*
>
> *Students: "Fran will be 12, and Baby will be 4."*
>
> *(The teacher put the numbers in the blanks as illustrated in Figure 11).*

Figure 11

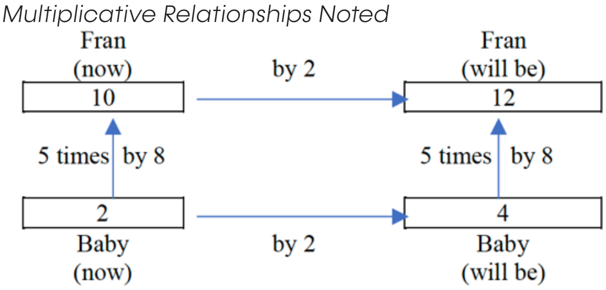
Teacher's Variation Problem: Expanded Version



> *Teacher: "Can you tell me the relationship between Fran will be, and Baby will be?*
>
> *Students: "It will be the same."*
>
> *(The teacher put both – "by 8" and "5 times" – in the schematic as illustrated in Figure 12).*

Figure 12

Teacher's Expanded Version with Additive and Multiplicative Relationships Noted



Jamie: "Wait a minute. 12 minus 4 is 8, but 12 divided by 4 is 3, not 5..."

Chris: "It is another trap!"

(But this time, nobody agreed with Chris).

Jordan: "No, it is not a trap."

Students talked about this situation for a while. Eventually, they concluded that the age difference was the same, but it was not true for the times relationship. They returned to the original problem. Students said, "The age difference of 24 is the same every year, but the "times" relationship is not like that.

Interpretations of Class Episodes

This section revisits each vignette and comments on several key points by juxtaposing the descriptive data from class episodes with our interpretations based on what we learned from other studies.

Comments on Vignette 1: Smooth Beginning, Doing Unasked Work, and a Trap!

Initially, the solution process was smooth. Examining the given multiplicative relationship between the father's age (now) and son's age (now), students identified the son's age (will be). Students then noted the additive relationships between current ages and ages in 4 years, which should have a 4-year difference. Students' recognition of these relationships resulted in incorrect answers. Jordan's proposal to further investigate the relationship between the father's and the son's ages in 4 years completely changed the way of the discussion. Two aspects were notable in terms of students' attitudes toward problem-solving. First, students had discussions on doing unasked work. Although there was some resistance to doing extra work, there was a consensus that they could do it. Rather than using the "keyword method" or adopting "number grabbing" approach (e.g., Littlefield & Reiser, 1993), being curious about unasked questions helped focus more on the structure of the problem context. Second, although there was no valid conclusion, the students acknowledged the possibility that the problem may have incomplete or contradictory information. In the end, it could not be solved (i.e., a trap problem). These two aspects show different attitudes toward problem-solving from what Schwieger (1999) points out as reasons for students' difficulties with problem-solving: (a) unwillingness to bring anything additional to the problem-solving process other than one numerical, symbolic answer and (b) belief of a singular solution and method for problem-solving.

While students showed desirable attitudes toward problem-solving, this session ended with an incorrect mathematical conclusion. The entire group agreed upon the erroneous conclusion that the multiplicative relationship ("times" relationship in students' words) between the father's and son's ages would remain the same in 4 years. Two aspects are noteworthy. First, the students were unable to identify the additive relationship between the father's age (now) and son's age (now) bound by the given multiplicative relationship between these two quantities.

Second, the students seemed to overgeneralize the invariance of relationships. In previous experiences, the students only examined problems involving the additive or multiplicative relationships; thus, the invariance of difference or ratio was preserved. In the problem context reported in this study, the students had to think about both additive and multiplicative relationships. Perhaps, their past learning experiences in techniques and language patterns triggered this overgeneralized conclusion (Mason, 2003). What happened in this vignette is evidence of students' unstable understanding of multiple relationships in one context.

Comments on Vignette 2: False Attention to the Shape of the Schematic Representations

In this vignette, Morgan attempted to prove that the problem was not a trap problem because there should be no relationship between the father's age and the son's age in 4 years. In Vignette 1, students' discussion was about "we don't need to do it" vs. "we still can figure it out." In this vignette, Morgan tried to explain that it was not possible to find the relationship between ages in 4 years by changing the shape of the schematic representations. Morgan attempted to separate these two quantities as much as possible, thinking that students could not "connect" them in the representations, and thus there should be no relationship. Other students' reactions to Morgan's idea (i.e., students were able to connect two quantities, regardless of the shapes Morgan created) eventually persuaded Morgan. Still, the idea that the problem itself was a trap remained.

Morgan's inaccurate attention to the shape of the schematic representations revealed the students' potential confusion, which had not been surfaced before. At the same time, the overall discussion in Vignette 2 showed the flexibility of the students in using schematic representations. Ng and Lee (2009) reported in their study that some students seem to treat the schematic models as an algorithm. However, it was not an algorithm learned by rote; rather, it was a problem-solving heuristic that required students to reflect on accurately depicting the information presented in word problems. Similarly, in this vignette, Morgan's false attention provided an opportunity to demonstrate students' cognitive flexibility in using schematic representations.

Comments on Vignette 3: Reconstructing the Problem

Noting the importance of distinguishing between additive and multiplicative situations, Mason (2018) suggested that encouraging students to make connections or develop problems with given structural relations would be an important area for further exploration. Students' discussion in Vignette 3 appeared to align with this suggestion. Alex's attempt to restructure the problem using different quantities helped students attend to the additive relationship. Alex focused on the difference between the ages because it was not immediately feasible to find the multiplicative relationship between the selected two quantities (32 and 11). At this point, the teacher intentionally suggested two quantities (10 and 2) so that the students could identify both additive and multiplicative relationships. Alex's and the teacher's attempts resonate with the notion of variation in structuring sense-making regarding tasks (Watson & Mason, 2006). Both aimed to expose the target mathematical structure by strategically varying some features of the problem while keeping other features. Reconstructing the schematic representations using these quantities (i.e., strategic variation) promoted students' focus of attention and encouraged them to notice what was invariant in this context. The students' willingness to reconstruct the context with the teacher's purposeful support was helpful.

Discussion and Implications

The story problem used in this article can be quickly solved using several steps of analysis and calculation. However, exploring this word problem with schematic representations took an unexpected path, resulting in a much longer exploration than expected. Some may say that this is a failure of lesson planning and its enactment. Others may question whether it was worth spending a long time discussing only one problem. While admitting that the presented class episodes in this study were atypical in terms of the duration of the discussion, we saw the value of allowing such an atypical learning process to occur.

Regarding the mathematical content, the students' lengthy investigation was fueled by their initial confusion and curiosity about the additive and multiplicative relationships and the related invariant and variant relationships. Considering the importance of constructing multiplicative reasoning for students' learning of mathematics throughout the middle grades and beyond (Zwanch & Wilkins, 2021), this was a timely opportunity for students to think about different relationships among quantities. Although it took longer than planned, it was worthwhile because it offered students a space to express their confusion, demonstrate their knowledge, test conjectures, construct a similar but different problem context, and eventually recognize multiple relationships within the particular problem context and general contexts. Additionally, the exploration revealed students' unexamined assumptions about the use of schematic representations.

We particularly noted that there were several instances where students themselves exhibited intellectual perturbations (Harel et al., 2014) or cognitive conflicts. Without such student-generated perturbations and conflicts, the proposed problem might have ended up as a computational problem. What if Jordan did not ask to find the unasked question in Vignette 1? What if Morgan did not pay her false attention to the shape of the schematic representations in Vignette 2? What if Alex did not suggest restructuring the problem using different examples? What if the teacher did not provide strategic variation in the quantities to shift students' attention? Such unexpected questions helped the students focus on the structures and relationships rather than just performing calculations.
Consistent with previous studies (e.g., Lehrer et al., 2000; Terwel et al., 2009), the findings of this study revealed that students could resolve their confusion through collaboration. While the teacher did not ask students to use a particular schematic representation, they constructed and reconstructed the representations through group discussion to reveal the difference between the father's and son's ages concerning additive and multiplicative relationships. Therefore, students could identify important mathematical elements in additive and multiplicative word problems and explain structural relationship of these problems using schematic representations. These findings revealed that teachers might provide students with mathematical tools to support their investigation, reasoning, and justification.

These findings also highlighted the teachers' roles in solving complex word problems with representations. Using mathematical tools, such as representations, alone could not guarantee students' mathematical learning (Lehrer et al., 2000). As Mason (2018) claimed "what seems to matter most is not the apparatus itself," (p. 332) but how teachers and students use them. If the apparatus is not used properly, its use might lead to rote learning. Therefore, teachers should be cautious when using schematic representations in mathematics classrooms. For example, as shown in this study, teachers could first teach their student types of schematic representations that they could use and explain the meanings of individual representations. Next, teachers could provide challenging problems and ask students to justify their reasoning by presenting additional questions. These processes might arouse students' curiosity and help them manipulate the quantities to reveal the mathematical structure of the problem.

As such, teachers should create a mathematics learning environment that allows students to investigate, reason, and justify (Depaepe et al., 2010; NCTM, 2000). As active investigators, teachers should believe in their students' mathematical abilities and refrain from transmitting mathematical knowledge and algorithms (NCTM, 2014). When teachers consider their students as passive listeners, students are unlikely to present unasked questions and investigate them. Moreover, teachers should respect students' authority in learning mathematics and create a classroom culture where all students' answers are respected (Cobb & Hodge, 2011). While some students gave incorrect answers in this study, most students did not criticize their ideas, and the teacher did not directly correct them. Instead, the students attempted to justify their arguments using representations and discussion, while the teacher supported their argumentation. Therefore, mathematics educators should be concerned with their classroom culture, particularly whether it facilitates or hinders students' understanding of mathematical relationships and structures in learning mathematics (Davydov, 1990; Mason, 2003; Zwanch & Wilkins, 2021).

The study has some limitations. Given that this study examined a small group of students in a single classroom, the findings of the study could not be generalized to other contexts. Therefore, studies with larger samples might yield more generalizable results. However, we hope that these classroom episodes give teachers and teacher educators ideas to explore more optimal learning environments for students to raise awareness about the mathematical structure and relationships in solving word problems.

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Investigation of the Relationship between Academic Competencies and Social Information Processing of 60–72 Month-Old Children

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Abstract

Starting from the preschool period, children need to grow up as individuals with high academic skills, academic enablers and respond positively to their social situations. Academic skills and academic enablers together constitute academic competence. The positive reaction of children to the problems they face constitutes social information processing. This study aimed to examine the relationship between the academic competencies of 60–72-month-old children and their social information processing. The study was designed with the relational survey method. The study group consisted of 132 children aged 60–72 months with normal development who attend preschool education. The data collection tools of the study are as follows: Personal Information Form, The Social Information Processing Interview–Preschool Version, and Teacher Rating Scales of Early Academic Competence. Spearman's rank-order correlation test was used to evaluate the relationship between the scales. The findings of the study revealed that there is a relationship between the interpretation of cues and response decision, which are subdimensions of the social information processing model, academic skills (numeracy, early literacy, thinking skills, and comprehension) and academic enablers (social-emotional competence, approaches to learning, and communication).

Keywords:

Academic Skills, Academic Enablers, Preschool Children, Social Information Processing

Introduction

Nowadays, in order for children to be successful and happy in their adulthood, it is necessary to raise children having high academic success, develop social skills, and respond positively to the problem situations they encounter, starting from the preschool period. Academic achievement is possible through the combination of academic skills and academic enablers toward acquiring academic skills, that is, by providing academic competence. Social information processing mechanism, which is responsible for the processing of new social information, forms the basis of the social situations that children encounter in their daily lives. Children's academic competencies and social information processing develop together.

Academic competence is the belief that children can perform the necessary actions for an academic task or activity (Duchesne & Larose, 2018; Niemiec & Ryan, 2009). The concept of competence, which plays a key role in achieving academic success, forms the basis of personal and environmental factors (Anderman & Patrick, 2012). The concept of academic competence consists of academic skills and academic enablers. Academic skills include numeracy, early literacy, thinking, and comprehension skills. Acquiring academic skills from the preschool period is important because it increases children's readiness for school and academic achievement in the following years (Reid et al., 2014). In addition, considering children constantly need such skills in their daily lives, they should be supported from the preschool period.

Mathematics skills are an important predictor of academic success (Cohrssen & Niklas, 2019). Mathematics skills make significant contributions to thinking and reasoning skills (Knowles, 2009). According to Stipek et al. (2001), mathematics is expressed as 'a static body of knowledge that includes a set of rules and procedures applied to give a single correct answer'. However, the development of early mathematics skills in the preschool period is the understanding of mathematical rules and procedures through various activities such as games and drama (Stipek, 2013). Early mathematical skills gained in this way include understanding mathematical symbols, relationships, comparisons (Wakabayashi et al., 2020), numbering, relations, and arithmetic operations (National Research Council, 2009). Early math skills are also referred to as early numeracy skills in the literature. Broadly examined, early numeracy skills include understanding and manipulating both symbolic and non-symbolic numbers (Raghubar & Barnes, 2017).

Symbolic number skills are associated with the development of counting skills and the development of numeracy skills. Early symbolic number skills include counting sequence, numerical meanings of numbers, and the last number indicates the number of objects in the group when counting a group of objects (Gobel et al., 2014; Merkley & Ansari, 2016). Studies have concluded that the acquisition of early symbolic number skills in the preschool period significantly affects mathematics achievement in the first grade of primary school (Gobel et al., 2014; Jordan et al., 2009; Jordan et al., 2007).

Non-symbolic number skills include numerical representations, relationships and comparisons without symbols (Raghubar & Barnes, 2017). Early non-symbolic number skills include adding and subtracting with three-dimensional objects and pictures, comparing the concept of quantity (such as telling which sequence is more), and making one-to-one matching (Leibovich & Ansari, 2016; Bisanz et al., 2005). Acquiring non-symbolic number skills is the basis of early numeracy skills. Studies have shown that non-symbolic number skills affect the continuity of subsequent mathematics performance (Leibovich & Ansari, 2016; Purpura & Logan, 2015). Early literacy is the acquisition of knowledge, skills, and attitudes, which are the prerequisites for children to learn how to read and write. Mathematical skills acquired in preschool period, mathematics knowledge in primary school period, and early literacy skills form the basis of literacy skills (Sonnenschein et al., 2021). As children's interactions with the environment increase, their critical and creative thinking skills increase, and children interpret, analyze, and evaluate what they live and learn based on their experiences (Pasquinelli et al., 2021). Comprehension is one of the basic academic skills that progress with children's basic language skills. Children with developed comprehension skills should better analyze and make sense of their academic skills (Kargın et al., 2017a).

Mathematical skills acquired in pre-school period constitute the basis of mathematics knowledge and early literacy skills in primary school period (Sonnenschein et al., 2021). In this direction, early literacy is the acquisition of knowledge, skills, and attitudes, which are the prerequisites for children to learn how to read and write, which are the prerequisites that children should acquire before learning to read and write. Early literacy skills are classified as verbal language, alphabet and letter knowledge, phonological awareness and print awareness (Elliott & Olliff, 2008; Kargın et al., 2017b).

As children's interactions with the environment increase, their critical and creative thinking skills increase, and children interpret, analyze, and evaluate what they live and learn based on their experiences (Pasquinelli et al., 2021). Critical and creative thinking skills are considered within the scope of high-level thinking skills. These skills are accepted as one of the lifelong learning processes. The acquisition of these skills from the pre-school period increases the developmental levels of children (Nachiappan, et al. 2019; Wojciehowski & Ernst, 2018). Comprehension is one of the basic academic skills that progress with children's basic language skills. Children with developed comprehension skills should better analyze and make sense of their academic skills (Kargın et al., 2017a).

Early academic enablers consist of learning approaches, social and emotional competence, fine motor skills, gross motor skills, and communication skills. Academic enablers were associated with academic skills, and it has been stated that children with positive attitudes toward academic achievement had high academic skills (Reid et al., 2014). Approaches to learning are the behaviors of children during acquiring academic skills (Bulotsky-Shearer et al., 2011).

When children's social-emotional competencies and communication skills are high, they have fewer problems in the classroom and can focus on academic skills. Therefore, children with high social-emotional competence and communication skills have a positive attitude toward acquiring academic skills (Denham & Bassett, 2020). Fine and motor skills are also effective on academic enablers, as they are related to children reaching sufficient maturity to perform academic skills (Cameron et al., 2016). The fact that children have high academic enablers ensures that they are more advanced in academic skills and that children's academic competence is high. Children with high academic competence behave determinedly in the educational environment, approach social situations more positively, and are highly motivated. Children with low academic competence are stressed in the educational environment, act reluctantly, and have low motivation (Schunk & Pajares, 2016). These explanations show that non-cognitive factors also focused on academic competence (Anthony & DiPerna, 2018; Duckworth & Yeager, 2015).

A framework including academic skills and behaviors, academic enablers, and social skills has been developed to define non-cognitive factors (Farrington et al., 2012). Children with higher academic competence respond positively to social situations they encounter (Konold & Pianta, 2005). In this direction, it is thought that there is a relationship between academic competence and the social information processing that explains children's responses, and the social information processing model is included.

The social information processing model relates beliefs, emotions, attributions, and responses (Larkin et al., 2013). According to this cyclical model, the formation of a behavioral response consists of a six-step cognitive process (Crick & Dodge, 1994). Individuals encode and interpret social cues in the first two steps before a response occurs in the face of social situations, they develop solutions for the situation they encounter with the clues they encode in the third step, they search for the appropriate behavioral response from their memories in the fourth step, they decide on the appropriate response in the fifth step and evaluate their responses in the sixth step (Zajenkowska et al., 2021; Ziv & Elizarov, 2020). Social information processing is linked to behaviors deemed appropriate by society (Mayeux & Cillessen, 2003). A positive reaction to the situations one encounters is socially acceptable behavior. Cognitive processes should be developed for the development of socially accepted and non-aggressive behaviors in the preschool period (Bierman et al., 2009). Because children who react positively are accepted by their peers, and their relationship with their teachers and peers develops. Thus, children's commitment to school increases, they are more motivated to learn what is taught at school, have a positive attitude toward learning, and increase their academic success increases (Konold & Pianta, 2005).

The importance given to children's growing up as adults who respond positively to the social problems they encounter has begun to increase (Şenol & Metin, 2021). Bierman et al. (2009) found that vocabulary, intelligence scores, participation levels, academic knowledge, and skills of the children who showed responses and behaviors that were not accepted by society were low. Studies showed that the academic competencies of children who have developed social competence in the society they live in have better academic competencies (Backer-Grøndahl et al., 2019; Franco et al., 2017; Ziv, 2013). In this direction, there may be a relationship between social information processing, which is emphasized to be effective on social skills and social competence, and academic competence. Responding to socially accepted behaviors within the scope of social information processing could lead to classroom harmony and more effective participation in activities. Thus, while contributing to forming a positive attitude toward learning, having negative responses creates a risk profile for academic competence.

It is thought that positive responses and behaviors accepted by society are associated with high social information processing and affect the development of academic competence. This effect also emerges from the preschool period. In this direction, the study aimed to examine the relationship between the social information processing of 60–72-month-old children and their academic competencies.

For this purpose, the study seeks answers to the following research questions:

- What are the social information processing and academic competence levels of 60–72-month-old children?
- Is there a relationship between social information processing and academic competence of 60–72-month-old children?

Method

Research Design

The study was designed with the relational screening model, one of the quantitative research methods. In relational screening models, the relationship between two variables is examined without any intervention (Büyüköztürk et al., 2016). This study was designed through a relational screening method as the relationship between social information processing and academic competence was examined.

Study Group

The study group consists of 132 60–72-month-old children with normal development who attended kindergarten and nursery schools in the spring semester of the 2020–2021 academic year.

Table 1
Demographics of Children Included in the Study

Demographics		*N*		%	
Gender	Girl	70		53.03	
	Boy	62		46.97	
	Total	132		100	
Children level	First child	35		26.51	
	Second child	43		32.58	
	Third child	44		33.33	
	Fourth child	10		7.58	
	Total	132		100	
		Mothers		Fathers	
		N	%	*N*	%
Education level	Second	28	21.21	12	9.09
	High	37	28.03	40	30.30
	University	67	50.76	80	60.61
	Total	132	100	132	100
Occupation	Unemployed	47	35.61	3	2.27
	Civil servants	33	25.0	58	43.94
	Workers	40	30.30	42	31.82
	Self-employed	12	9.09	29	21.97
	Total	132	100	132	100

The demographic characteristics of the children in the study group show that 53.03% of the children are girls, and 46.97% are boys. When the birth order of the children was examined, it was seen that 26.51% were the first, 32.58% were the second, 33.33% were the third, and 7.58% were the fourth. About 21% of the children's mothers are secondary school graduates, 28.03% are high school graduates, and 50.76% are university graduates. Approximately 9% of their fathers were secondary school graduates, 30.30% are high school graduates, and 60.61% are university graduates. Twenty-five percent of mothers are civil servants, 30.30% are workers, 9.09% are self-employed, and 35.61% are unemployed. Almost half of the fathers (43.94%) are civil servants, 31.82% are workers, 21.97% are self-employed, and %2.27 are unemployed.

Data Collection Tools

Personal Information Form, Teacher Rating Scales of Early Academic Competence (TRS-EAC) and The Social Information Processing Interview–Preschool Version (SIPI-P) were used as data collection tools.

Personal Information Form

The form consists of questions about the children's gender, their ages, the number of siblings, the educational status of the parents, their profession, age, and income status of the family.

Teacher Rating Scales of Early Academic Competence (TRS-EAC)

TRS-EAC was developed by Reid et al. (2014) to measure the early academic competence of 38–70-month-old children, and it was adapted into Turkish by Şenol and Turan (2019). TRS-EAC consists of a combination of the following two subscales: Early Academic Skills (EAS) and Early Academic Enablers (EAE). The subscales consist of 35 and 46 items, respectively. The Early Academic Skills subscale consists of the following subdimensions: "Creative Thinking (CRT), Critical (CLT) Thinking Skills, Numeracy (N), Early Literacy (EA), Comprehension (C)." Early Academic Enablers subscale consists of the following subdimensions: "Approaches to Learning (AL), Social and Emotional Competence (SEC), Fine Motor Skills (FM), Gross Motor Skills (GM), and Communication (C)." Each statement about academic competence in the scale is scored as significantly below age expectations (1), below age expectations (2), compatible with age expectations (3), above age expectations (4), and significantly above age expectations (5). From the Early Academic Skills Scale, participants scored a minimum of 35 and a maximum of 175; participants obtained the lowest score of 46 and the highest score of 230 from the Early Academic Enablers Scale. The Cronbach alpha internal consistency coefficients of the Early Academic Skills Scale were found to be .98 and ranged from .94 to .97 for its subdimensions. The Cronbach's alpha internal consistency coefficients of the Early Academic Enablers Scale were found to be.98 and ranged from .89 to .97 for its subdimensions (Reid et al., 2014). In this study, the Cronbach's alpha internal consistency coefficient of the Early Academic Skills Scale was .96, and its subdimensions ranged from .97 to .98. The Cronbach alpha internal consistency coefficients of the Early Academic Enablers Scale were found to be .92 and ranged from .96 to .99 for its subdimensions.

The Social Information Processing Interview-Preschool Version (SIPI-P)

The social information processing interview-preschool version (SIPI-P) was developed by Ziv and Sorongon (2011) to obtain information about children's social information processing, and it was adapted into

Turkish by Şenol and Metin (2019).

The test consists of four stories. The first and third stories are about a child who is offended by a peer, while the second and fourth stories are about a child trying to participate in the game of his two playing peers. The test has separate forms for girls and boys. These forms include parallel pictures and the same stories. The stories were read to the children one-on-one by the researcher in a quiet setting, and children's answers to the questions about the story were written on the answer form. SIPI-P has three subdimensions Scores from SIPI-P are effective in predicting children's social behavior. The three subdimensions are interpretation of cues, response construction, and response decision. It was observed that the internal consistency coefficient of SIPI-P was .76 for the interpretation of cues sub-subdimension, .78 for the response construction subdimension, and .87 for the response decision subdimension (Ziv & Sorongon, 2011). The internal consistency coefficients calculated in this study were .70, .74, and .76, respectively.

Data Collection

Approval was obtained from the parents by interviewing the principals of the kindergartens to be implemented. Children whose parents gave consent were informed about the application, and children who volunteered were included in the study. Data were collected from children aged 60–72 months who received preschool education in the fall semester of the 2020–2021 academic year.

SIPI-P data were collected through reading stories and asking questions to children. The answers given by the children were written on the answer form. Data were collected through individual interviews with the children. The administration of the test took approximately 20–25 minutes for each child. Teachers filled out the TRS-EAC for the children one by one. It took approximately 20 minutes to complete the scale for each child.

Analysis of Data

Percentage and frequency of demographic information; continuous data were presented as mean, standard deviation, median, maximum, and minimum. The normal distribution properties of continuous data were evaluated using the Kolmogorov-Smirnov test, and the findings of the analysis revealed that it did not fit the normal distribution. Spearman's rank-order correlation test was used to evaluate the relationship between scales. Significance level in the study was set at $p < .05$.

Results

In this section, first, descriptive analysis of children's scores from the Teacher Rating Scales of Early Academic Competence (TRS-EAC) and The Social Information Processing Interview–Preschool Version (SIPI-P) and the analysis of the relationship between the two scales were discussed.

The mean scores of the children from the Academic Skills Sub-Scale were "numeracy skills (M = 18.11, SD = 3.49), early literacy (M = 18.27, SD = 3.90), creative

Table 2

The Mean Scores of the Children in TRS-EAC and SIPI-P

Scales		Subdimensions	Mean	Median	*SD*	Min.	Max.
TRS-EAC	Academic skills	Numeracy	18.11	19.00	3.49	10.00	25.00
		Early literacy	18.27	20.00	3.90	10.00	40.00
		Creative thinking	28.93	31.00	5.50	16.00	40.00
		Critical thinking	36.39	38.00	6.68	20.00	50.00
		Comprehension	25.65	28.00	4.69	14.00	35.00
		Total	127.36	128.50	23.32	70.00	175.00
	Academic enablers	Approaches to learning	53.87	55.50	10.92	0.00	75.00
		Social-emotional competence	43.75	46.00	8.69	16.00	60.00
		Communication	36.39	38.00	6.90	20.00	50.00
		Fine motor	21.68	23.00	4.18	12.00	30.00
		Gross motor	10.88	12.00	2.32	0.00	15.00
		Total	166.57	170.00	31.66	83.00	230.00
SIPI-P		Interpretation of cues	2.24	2.00	0.77	1.00	4.00
		Response construction	0.14	0.00	1.19	1.00	3.00
		Response decision	37.28	37.00	4.70	26.00	46.00

thinking (*M* = 28.93, *SD* = 5.50), critical thinking (*M* = 36.39, *SD* = 6.68), comprehension skills (*M* = 25.65, *SD* = 4.69), and total (*M* = 127.36, *SD* = 23.32)." The mean scores of children from the Academic Enablers Sub-Scale were "approach to learning (*M* = 53.87, *SD* = 10.92), social-emotional competence (*M* = 43.75, *SD* = 8.69), communication skills (*M* = 36.39, *SD* = 6.90), fine motor skills (*M* = 21.68, *SD* = 4.18), gross motor skills (*M* = 10.88, *SD* = 2.32), and total (*M* =166.57, *SD* = 31.66)." High scores from the Teacher Rating Scales of Early Academic Competence indicate high academic competence. These results were similar to the findings of studies using the same measurement tool (Reid et al., 2014; Sezgin & Ulus, 2020).

The mean scores of the children on the Social Information Processing Interview-Preschool Version were "interpretation of cues (*M* = 2.24, *SD* = 0.77), response construction (*M* = 0.14, *SD* = 1.19), and response decision (*M* = 37.28, *SD* = 4.70)". Higher scores of children in SIPI-P indicate more positive response decisions under the social information processing model. The results obtained from the subdimensions of interpretation of cues and response decision were similar to the results of studies using the same measurement tool (Şenol & Metin, 2021; Ziv, 2013; Ziv & Sorongon, 2011). No studies supported the low score obtained from the subdimension of the response construction. The reason for getting a low score on this subdimension was associated with the cultural characteristics of the study group.

As Table 3 shows, there was a statistically significant positive correlation between the total scores and subdimensions of the Teacher Rating Scales of Early Academic Competence and the subdimensions of the Social Information Processing Interview-Preschool Version. However, no relationship between the "response construction" subdimension of SIPI-P and TRS-EAC was observed. There was a correlation between interpretation of cues and numeracy (*r* = .174), early literacy (*r* = .223), creative thinking (*r* = .203), critical thinking (*r* = .209), comprehension (*r* = .198) academic skills sub-scale total (*r* = .218), communication (*r* = .203), fine motor (*r* = .220) and gross motor (*r* = .174) sub-dimensions. There was a correlation between response decision and numeracy (*r* = .318), early literacy (*r* = .255), creative thinking (*r* = .337), critical thinking (*r* = .275), comprehension (*r* = .311), academic skills sub-scale total (*r* = .310), approaches to learning (*r* = .257), social emotional competence (*r* = .304) communication (*r* = .330), fine motor (*r* = .295), gross motor (*r* = .233) and academic enablers sub-scale total (*r* = .273) sub-dimensions.

Table 3

Relationship between Early Academic Competence and Social Information Processing

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1. Numeracy	r	1.000														
2. Early literacy	r	.898**	1.000													
3. Creative thinking	r	.957**	.884**	1.000												
4. Critical thinking	r	.921**	.952**	.913**	1.000											
5. Comprehension	r	.944**	.936**	.930**	.941**	1.000										
6. Total (academic skills sub-scale)	r	.956**	.953**	.958**	.978**	.970**	1.000									
7. Approaches to learning	r	.918**	.915**	.910**	.921**	.934**	.936**	1.000								
8. Social emotional competence	r	.897**	.912**	.882**	.915**	.912**	.923**	.964**	1.000							
9. Communication	r	.925**	.940**	.930**	.942**	.937**	.961**	.926**	.939**	1.000						
10. Fine motor	r	.917**	.896**	.927**	.899**	.913**	.932**	.911**	.906**	.951**	1.000					
11. Gross motor	r	.895**	.873**	.892**	.882**	.891**	.900**	.940**	.909**	.895**	.888**	1.000				
12. Total (academic skills sub-scale)	r	.919**	.921**	.908**	.930**	.929**	.942**	.987**	.983**	.943**	.930**	.943**	1.000			
13. Interpretation of cues	r	.174*	.223*	.203*	.209*	.198*	.218*	.154	.143	.203*	.220*	.174*	.167	1.000		
14. Response construction	r	.021	.015	-.003	-.012	.041	.013	.006	-.010	.018	.004	-.007	.003	.070	1.000	
15. Response decision	r	.318**	.255**	.337**	.275**	.311**	.310**	.257**	.304**	.330**	.295**	.233**	.273**	.351**	.250**	1.000

***. Correlation is significant at the .01 level (2-tailed).*
**. Correlation is significant at the .05 level (2-tailed).*

Discussion

As a result of the study examining the relationship between the social information processing of 60–72-month-old children and their academic competence, a relationship was found between the subdimension of interpreting social cues and the decision to react and the subdimensions of the academic competence scale. The decision to interpret and respond to social cues is an indicator of whether children have a positive or negative response and intention to a situation. Similarly, children who react positively to their peers and maintain these behaviors have higher academic achievement and attitudes toward learning (Denham & Brown, 2010; Fantuzzo & McWayne, 2002). In addition, the academic achievement of children excluded by their peers and interacting negatively with them is low (DeRosier et al., 1994; Ladd & Coleman, 1997). In the society or class, they live in, the children's academic competence with higher social competence is in a better position than those with lower social competence (Backer-Grøndahl et al., 2019; Franco et al., 2017; Ziv, 2013). The results obtained from the study are explained below, taking into account the subscales and subdimensions.

The findings of the study revealed that there is a relationship between children's academic skills (numeracy, early literacy, and thinking skills) and the interpretation of cues, and the response decision. It is necessary for children to meet the world of mathematics in the preschool period for their success in mathematics. Children can code numbers from infancy (Cordes & Brannon, 2008). Contributing to their number skills from the pre-school period increases their numerical thinking skills (Jordan et al., 2006). There is evidence that mathematical skill can be developed from interactions with the physical, social and cultural worlds (Alibali & Nathan, 2012). The development of mathematical skills can positively improve not only mathematical problems but also solution methods for social problems. In this way, positive effects can occur on children's social competencies. Since studies have proven that there is a relationship between mathematical skills and social competence in early childhood, it is important to develop children's social competences in the development of early mathematics skills (Duncan & Magnuson, 2011; Griffin, 2004). Developing children's social competencies is important for developing early mathematics skills (Duncan & Magnuson, 2011). In a study on Latino children, it was stated that social competence plays an important role in the growth of preschool children's early mathematical skills (Galindo & Fuller, 2010). In addition, in a study examining the relationship between early cognitive skills and social competence, it was determined that social competence predicted early mathematics skills (Scott, et al., 2013). Contribution of social competence including interpersonal skills to early mathematics skills has been emphasized in studies (Ginsburg, 2006; Master et al., 2016). Responding positively to situations is the basis of social competence. The current study, in that sense, is consistent with the results of the study of Mackintosh and Rowe (2021) examining the role of preschool children's social problem-solving skills in the development of early mathematics skills. Similarly, studies reported that there is a relationship between preschool children's early math skills and social competencies (Dobbs et al., 2016; Doctoroff et al., 2016; Galindo & Fuller, 2010). In a study by Denham et al. (2012), it was found that children with low numeracy and early literacy skills displayed more aggressive behavior. Bierman et al. (2008) reported a relationship between children's social information processing and early literacy skills, and a relationship was found between ''adequate'' (prosocial or assertive) and ''inadequate'' (passive) behavioral solutions and literacy skills. In addition to numeracy and early literacy skills, critical and creative thinking skills are evaluated within the scope of academic skills. Social skills and social information processing of children with weak thinking skills are adversely affected (Fonagy et al., 2018; Ziv & Arbel, 2020). Studies had shown that when children's alternative thinking skills were developed, their social competencies also improved (Arda & Ocak, 2012), and they responded positively to social situations (Logie, 2014). The results of these studies on thinking skills were consistent with the results obtained from this study. The role of different developmental areas should be considered to facilitate the learning of academic skills in early childhood. Within the scope of the study, the fact that the social information processing steps are related to academic skills shows that the social information processing is effective in the acquisition of early academic skills. In other words, children with more competent social information processing may be more likely to acquire better academic skills. This result should be explained by the fact that children responding positively to social situations experience positive results in their social lives; therefore, it would be safe to say that children concentrate more on academic skills.

A relationship between children's early academic enablers (Learning Approaches, Social and Emotional Competence, Fine Motor Skills, Gross Motor Skills, and Communication), the interpretation of social cues, and the decision to react was observed. According to the study findings, early academic enablers are the behaviors of children during acquiring academic skills. It also reflects children's basic perceptions of learning. Therefore, it is safe to say that it is effective in the academic competence of children. The findings of the study revealed that there was a relationship between approaches to learning and social information processing. Similarly, some studies found a relationship between children's approaches to learning and problem behaviors (Bulotsky-Shearer et al., 2011; Escalon & Greenfield, 2009; Fantuzzo et al., 2005). In addition, the result obtained was compatible with the finding of the study of Ziv (2013), claiming that there was a direct relationship between social information processing and approaches to learning. Early academic enablers include children's social-emotional competencies and communication skills (Reid et al., 2014). The results of the current study show that there is a relationship between preschool children's social and emotional competence and communication skills and the social information processing. This result was compatible with the findings of Denham and Bassett's study (2020), reporting that there was a relationship between social competence and social information processing. Some studies claimed that children with high social-emotional competencies gave positive reactions to social situations and had positive social information processing structures (Denham et al., 2014; Nix et al., 2013). Studies examining communication skills reported that the communication skills of children who responded positively in social situations were high (Burks et al., 1999; Gifford-Smith & Rabiner, 2004). Children's negative reactions and less interaction with their peers can cause low communication skills. Being competent in motor skills is effective in academic enablers. According to the study findings, there is a relationship between motor skills and social information processing. Similar studies found a relationship between motor skills and social competence (Giske et al., 2018; You et al., 2019). Another study also found that children's motor skills positively affected their social competence (Özkara & Kalkavan, 2021). It was noteworthy that there was a relationship between attitudes that provide academic success and social information processing. The results showed that children's academic achievement attitudes are associated with socially competent mental representations when encountering social situations. In the process of social information processing, children who respond positively to social situations spend less mental energy in social situations, thus enabling early academic enablers to rise.

The study found no relationship between the Teacher Rating Scales of Early Academic Competence's subscales and subdimensions and the "Response Construction" subdimension of the Social Information Processing Interview-Preschool Version. In the response construction subdimension, children interpret the clues in the previous step and diversify their responses. Rising scores in this subdimension indicate that children form positive responses. All steps of the Social Information Processing Model are interconnected and have a cyclical structure. While a relationship between other subdimensions and academic efficacy was observed, it is noteworthy that these responses are not found in the subdimension of diversification. This may be due to the social and cultural characteristics of the study group.

Conclusion

The study found a relationship between the academic competence of 60–72-month-old children and the decision to interpret cues and react, which are subdimensions of the social information processing. This result can be interpreted as the higher the academic competence of children, the more competent they may be in social information processing. Children's positive perception and positive response to the social situations they encounter from preschool play an important role in providing and supporting academic competence and increase the social information processing of children.

Limitations and Recommendations

- This study examined the relationship between children's academic competencies and social information processing. Further experimental studies should be conducted to reveal the reason for this relationship.
- The study did not utilize any sample selection therefore; future studies can adopt probabilistic sampling method in selecting the study group to ensure the generalizability of the results.
- Further studies can evaluate the social information processing and academic competence in terms of demographic variables.
- Longitudinal studies should be conducted to determine the relationship between academic competence and social information processing in the primary school period.

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How Do We Learn Mathematics? A Framework for a Theoretical and Practical Model

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Abstract

The purpose of this paper is to propose an effective learning environment for the initial stages of mathematical learning. Basic numerical skills and the objects and actions that trigger those skills are conceptualized as a mathematics-learning environment. We discuss numerical learning mechanism and the basic skills and environments we use to learn numbers briefly within the human cognition system. The three subsystems of number, i.e., exact number system, approximate number system and access to symbol system, are explained with reference to basic number competencies. They are discussed within the framework of "number sense" by drawing evidences from the neuroscience and mathematics education literature. Finally, how to manipulate the components of these subsystems for an effective learning of number is exemplified in a proposed model of mathematical learning environment.

Keywords:

Number Sense, Core Systems Of Number, Quantification, Magnitude, Quantity

Introduction

A In this paper, human cognition system, numerical learning mechanisms and the basic skills and environments we use to learn numbers are discussed within the framework of "number sense". Basic numerical skills and the objects and actions that trigger those skills are conceptualized as the parameters of mathematics-learning environment. First, let us explain some basic concepts used in this paper for a better understanding of the proposed model.

Number Sense: It is the ability to use numbers intuitively, effectively, efficiently and fluently in problem situations.

Human cognition system: The theoretical structure that humans use to acquire any knowledge, skill or habit.

Basic number processing skills: It includes the perception of quantity, magnitude, approximate number estimation, and the ability to establish a symbol quantity-magnitude relationships, which are also known as core skills that enable people to learn mathematics.

Quantity: Quantity is the amount of something, which

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might be either discrete or continuous. While we call countable quantities as discrete quantities, we call continuous quantities as magnitudes. We use somewhat different actions to quantify a magnitude or a discrete quantity. For example, magnitudes can be measured or estimated, while a discrete quantity can be counted or estimated. Discrete quantities less than 5 are called small, because they can be perceived at a glance through parallel processing, a.k.a., subitizing (Mandler & Shebo, 1982). Discrete quantities greater than 5 are called large since they can be enumerated either by guessing or by counting and/or calculations. As counting methods evolve, arithmetic and other computational methods such as using facts, skip counting, repeated addition, and multiplication are applied on them.

Reaction and response time: The time elapsed between seeing the task and answering it is called reaction or response time. In some studies, reaction time and response time are separated. Special experimental setups are required for this distinction. In this paper, we use the combination and call it reaction time.

Canonic arrangement: The countable quantities are arranged in a way that creates a pattern such as a dice pattern that facilitates perception (Piazza, Mechelli, Butterworth, & Price, 2002).

Random arrangement: The countable quantities are scattered or randomly arranged without any recognizable pattern that facilitates perception.

What is Number Sense?

As defined above, number sense is "the ability to intuitively use numbers effectively, efficiently and fluently in problem situations". Although a uniform number sense is mentioned in this definition, we can deduce that it may have different reflections as there will be changes in the concept and types of number at various age and grade levels, and therefore different measurements should also be required. For example, while for a kindergarten or first grader it is meant to use natural numbers fluently from one to ten or one to twenty, it may extend up to 100 or 1000, for a middle school student, on the other hand, this concept naturally includes fractions, decimals and arithmetic facts. In more advanced grades, however, we see that these basic skills are transformed into the use of, for example, algebraic expressions while making transformations, simplifications, and expansions. Therefore, these can also be considered as further extensions of the sense of number.

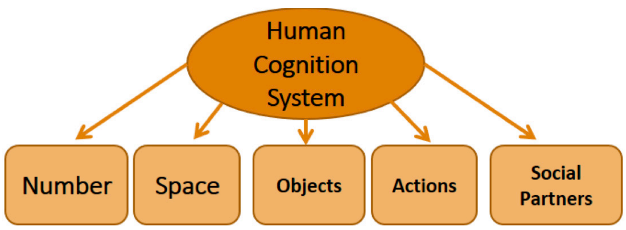
In the following sub-headings, the development of the concept of number in humans from birth will be discussed and the core knowledge, basic number processing systems and skills that enable further numerical learning will be elaborated. Basic numerical skills include the perception of quantity, the relative size and the place of number on a number line, its neighbors, their size relationships with other numbers, and the representation of numbers with symbols. As in all kinds of learning, there is a cognition system for people to learn numerical concepts and relations. This system as a whole mediates the learning, but there are also specialized subsystems for different aspects of numbers. Let us now consider them in detail.

Human Cognition System

As shown in Figure 1, the human cognition system consists of a small number of (4 or 5) subsystems (Kinzler & Spelke, 2007). These subsystems are cognitive structures used to represent number, space, objects, actions, and the social environment. It is believed that human beings are born in a way that is programmed to mentally represent the regularities they experience in their environment (Dehaene, 2009), we simply call this learning. The human species achieves this action through these specialized subsystems and networks between them. There are also evidence that some of these subsystems exist in some animal species (Spelke, 2017). However, the cognition system in humans is more comprehensive and complex than the one in other animals. This allows people to learn knowledge that is more abstract as well.

While the human cognition system represents quantity with the number subsystem, it represents shape and space with the space subsystem. It is the task of the objects subsystem to be able to think of one object as separate from another object, and learn the properties of each object building on this core knowledge. The task of imagining and thinking that a moving thing is moving independently from the object falls under the expertise of the actions subsystem. The social environment subsystem, on the other hand, is mostly reserved for the representation of subjects such as language, kinship, and cultural accumulations. Since the focus of this paper is on learning mathematics, the number subsystem and its constituent structures will be emphasized. A more elaborate discussion of the human cognition system can be found in (Kinzler & Spelke, 2007; Spelke & Kinzler, 2007).

Figure 1
Human cognition system

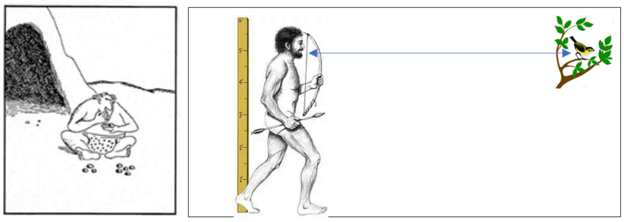


Source: Spelke and Kinzler (2007) Developmental Science 10:1, ss. 89–96.

Number Subsystem

The number subsystem is built on quantity perception. The amount appears in two different ways. These are "discrete quantity", that is, countable quantity, and "magnitude", that is, continuous or measurable quantity. Humankind has evolved this system, which it shares with some other animal species, and has created more useful forms for its life (Dehaene, Molko, Cohen, & Wilson, 2004). It is claimed that one of the reasons for mathematics learning disability or dyscalculia may be problems in quantity perception (Mazzocco, Feigenson, & Halberda, 2011). Each type of quantity triggers different mathematical actions and processes. Let us now examine these quantity types and their special cases.

Figure 2
Discrete and continuous quantity in primitive times

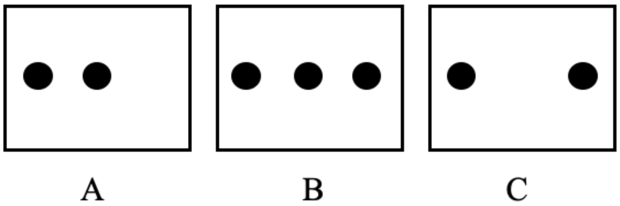


Discrete Quantity

Sets of objects that are countable are called discrete quantity. There is a difference in representation and perception depending on whether the quantity is small or large. While we call the quantity whose number we can perceive at a glance as small quantity, we call the quantity that exceeds this limit large quantity. The border between less and more is 4 or 5, also known as the subitizing range (Mandler & Shebo, 1982). When the number of objects is less than five, the human brain can perceive this number at a glance through parallel processing. When the number of objects exceeds five, and if there is no special arrangement, it cannot be detected at a glance, and other actions take place instead. While we use subitizing to enumerate small sets of objects rapidly, counting and/or other calculation operations are needed to enumerate larger sets.

There are evidences that the mechanism of perceiving small quantities is present from birth. In an experiment conducted by Antell and Keating (1983), it was revealed that 7-day-old infants were able to distinguish small quantities from each other, i.e., one from two or two from three. In this experiment, which was carried out using the looking time paradigm, the first group of infants was shown the card A consisting of 2 objects, and after the infants' attention was distracted from the card (practice, habituation), this time the card B consisting of 3 objects was shown (see Figure 3). The infants in the second group were shown the card C after the card A. It was found that the infants in the first group looked longer at the second card. This was shown as evidence that 7-day-old infants noticed the numerical differences in these cards. It was claimed that babies who did not even know number words or even speak yet use a kind of visual perceptual mechanisms to make this distinction.

Figure 3
An experiment with seven days-old infants



Again, in many experiments with adults, it was found that the responses to small numbers of objects less than five and large numbers more than five were different. Some researchers (Balakrishnan & Ashby, 1992) claimed that they did not find any evidence showing that subitizing is a separate mechanism. On the other hand, many other researchers(Benoit, Lehalle, & Jouen, 2004; Clements, Sarama, & MacDonald, 2019; Desoete, Ceulemans, Roeyers, & Huylebroeck, 2009; Piazza et al., 2002; Schleifer & Landerl, 2011) suggest that there is a different mechanism for perceiving small quantities of less than 5 and that it could act as a stepping stone for learning the cardinal number value and arithmetic facts.

In their study, Olkun, Altun, and Göçer-Şahin (2015) found that primary school children spent almost the same amount of time enumerating three and 4 dots. They were even relatively faster in counting four items. This may be because arrays of four objects are easier to perceive than three objects. After the number of dots exceeds four, not only the response time increases in parallel with the number of objects but also the gap between low-achieving and high-achieving students widens (see Figure 4). Other researchers also found discontinuity between subitizing and counting for dyscalculic children (Schleifer & Landerl, 2011). Another noteworthy detail in Figure 4 is that all groups, except the dyscalculia risk group, were faster in determining the canonically arranged eight dots compared to seven dots. Similar results that spatial arrangements of objects affected enumeration was also reported in the literature (Piazza et al., 2002). This finding also shows that canonically arranged dots facilitate perception and provide the opportunity to use different mental actions such as faster enumeration strategies. In fact, to support this argument, Piazza et al. (2002) claimed that subitizing and counting triggered different neural mechanisms.

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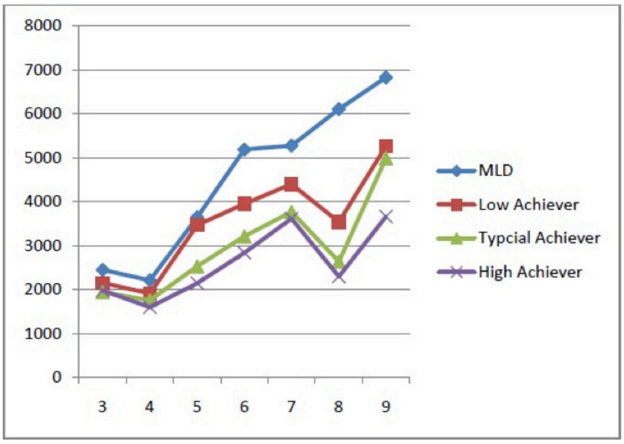
Figure 4

The medians of counting the canonically arranged quantities of 3-4-5-6-7-8 and 9 dots according to the achievement groups of the 2nd grade students



Source: (Olkun, Altun, & Göçer-Şahin, 2015)

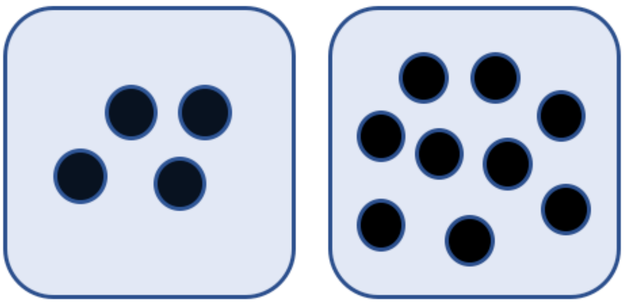
Many studies (Butterworth & Laurillard, 2010; Olkun, Altun, Göçer Şahin, & Akkurt Denizli, 2015) have found strong relationships between quantity perception and mathematics achievement. It has even been claimed that malfunctions in the quantity perception system can be a potential screening tool for mathematics learning disability (Desoete et al., 2009). Now let us do an experiment together to understand the difference in large and small quantity perception. You can repeat the experiment with different number of dots and with different people so that you can experience more reliable information first-hand.

Experiment 1.

Cover both of the quantities in Figure 5 with one hand each. Get a friend across you and quickly (in less than a second) open one hand and close it back. Ask how many dots there are. Then quickly open and close your other hand. Ask how many dots there are. Evaluate your friend's answers. Which one did s/he answer more correctly? In which one did s/he say a close number? Ask what actions s/he used to enumerate each "quantity".

Figure 5

Small number and large number



Another study examined whether 6-month-old infants (Wynn, 1992) understood the consequences of simple arithmetic actions. It was found that infants noticed when a new object was secretly added or removed from a small set of (>4) objects and showed a longer reaction time that could be regarded as astonishment to the incorrectly displayed result. For example, they responded with surprise that when two objects were shown and one object was added to it behind the scenes, the result was shown as two. However, the same infants remained indifferent when the number of objects treated was four or more. These experiments show that the mechanism of dealing with small numbers is present at very young ages, perhaps with birth, and the same system continues to be used in some form in adulthood. Some researchers tried to replicate the Wynn's study but found little or no evidence that infants can do addition or subtraction (Wakeley, Rivera, & Langer, 2000). It was concluded that simple adding and subtracting develops gradually throughout infancy and early childhood.

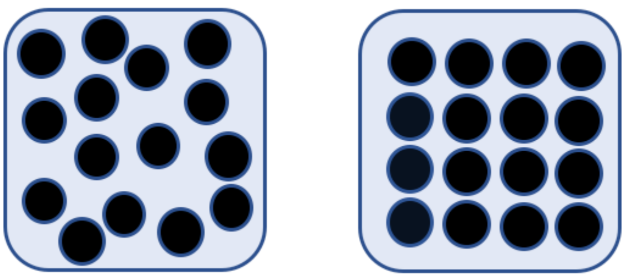
While trying to determine the numerosity of a "quantity", one of the factors affecting this is the arrangement of the objects that make up the quantity (Benoit et al., 2004). Differently arrayed objects trigger different actions, and different actions can reveal different mathematical processes (Olkun, Karslı-Çalamak, Sözen-Özdoğan, Solmaz, & Haşlaman, 2018). To examine this situation, repeat the following experiment with a friend.

Experiment 2.

Cover both of the sets in Figure 6 with one hand each. Take a friend in front of you and quickly (about 1-1.5 seconds) uncover one hand and cover it back while you ask how many dots there are. Then quickly (approximately the same passage of time) open and close your other hand. Ask how many dots there are.

Figure 6

Random and canonically arranged discrete quantities



Evaluate your friend's answers. Which one did s/he answer more correctly? In which one did s/he say a close number? In both experiments, ask your friend what actions s/he used to determine the number of sets. If s/he finds it difficult to answer, you can show her/him to choose the action listed below.

List of actions in quantifying a quantity:

Counting: Counting objects one by one

Subitizing: Perception of the numbers of groups of less than five at a glance

Grouping: Seeing objects in perceptible small quantities at a glance

Calculation: Finding the total number of objects in groups by using number facts

Estimating: Approximating the quantity or magnitude

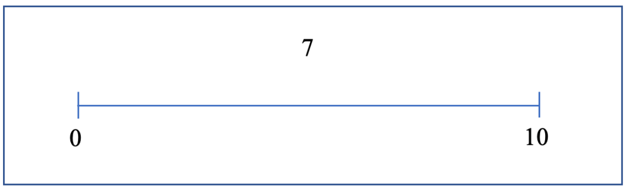
Measuring: Finding the size of a continuous quantity using a natural or a standard unit and unit iteration

These are the basic actions for quantifying a quantity; however, some combinations of the actions above might be used for enumerating large size quantities.

Magnitude

Another type of quantity that mathematics tries to quantify is magnitudes. Magnitude is also known as "continuous quantity". Concepts such as length, area, volume, and time are considered as continuous quantities. Continuous quantities, that is, magnitudes, trigger different actions and processes than that of discrete quantities. For example, while determining the number of a countable "quantity", it is necessary to use actions such as counting, grouping, calculating, estimating, however we use estimating or measuring for quantifying a continuous quantity. If we want to count or calculate continuous quantities, we must first make them countable by using a unit (hand span, meter, square unit, minute, hour, etc.).

Figure 7
A typical number line estimation task



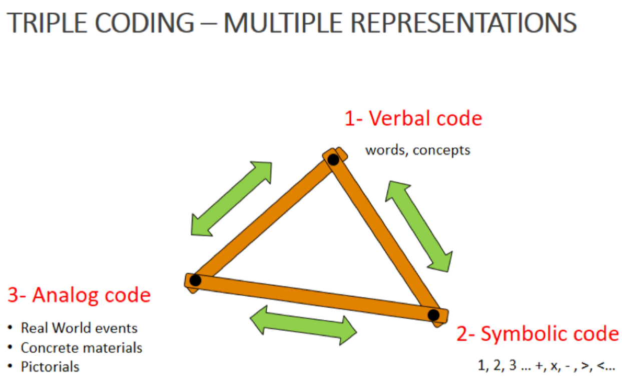
The most commonly used analog quantity in research and educational settings is the number line (Booth & Siegler, 2006). For example, a number line used for preschool and primary school first grade students is shown in Figure 7. By showing a number line, a child is asked "This number line has zero at the beginning and ten at the end. Where do you think seven is on this number line? Do you make a hash mark?" Thus, it is tested whether the child knows numerical concepts such as the reading, location, symbol, relative size, and positioning of numbers in the range of 0-10. Here, the child is expected to find the approximate location of the number rather than providing an exact hit. By finding the amount of error in the predictions made by the children, the estimation skills on the number line, in other words, the number sense skills are evaluated.

Access to Symbol System

Another task of the number subsystem is to establish a connection between Arabic number symbols and quantity. In other words, it is to be able to think of the symbol equivalent of the quantity shown, or vice versa. According to the triple coding theory (Dehaene, 1992) any mathematical knowledge is coded (represented) in three different codes or modalities. These are symbolic code, analog code and verbal (see Figure 8). This subsystem is also used to make transitions between the codes we use for a quantity fluently.

All kinds of concrete tools, drawings, graphics, or real-life situations are called analog representations. The word analog comes from the word analogy, which has been used to mean similarity. Its usage here means similar to the original event. In other words, quantity comes first either visible or hearable as in dram beats. That is, the amount is perceived first as analog quantity, and then this perception is converted into symbol(s) and word(s). In the future, it is constantly transcoded from one to the other.

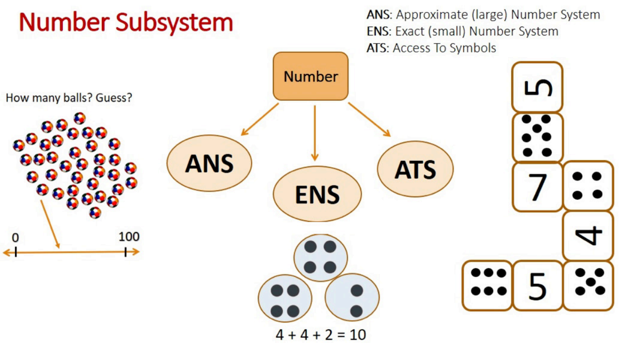
Figure 8
Triple coding or multiple representations of mathematical knowledge



It is claimed that the triple coding theory is also a suitable framework for examining performance in complex mathematical problem solving from neuro-psychological perspectives (Schmithorst & Brown, 2004). If we show the issues discussed so far regarding the number system on a diagram, we can say that the number system in human cognition consists of 3 sub-systems, and these are Approximate (or large) Number System (ANS), Exact (or small) Number System (ENS), and Access to Symbol System (ATS), (see Figure 9).

Figure 9

Subsystems of number in human cognition and sample tasks for measuring each subsystem



The time given for enumerating a "quantity" or "magnitude", whether presented as an analog quantity, as a symbol or verbally, is also affective on the action to be used for the quantification. For example, if the quantity is large and the given time is very short, a wild guess is used, in case the time increases a little, educated guessing based on the use of some strategies (i.e. estimation) can be used.

Summary and Conclusions

In the simplest terms, we can define learning as "recognizing or discovering the regularities in our experiences". We can define learning mathematics as noticing the patterns between numbers and shapes and expressing them with the language of mathematics. Expressing in the language of mathematics, or in short, mathematization, is the process of representing recognized patterns using numbers and other symbols, or transcoding between representations. We can think of a large part of the mathematization process as a quantification process.

As summarized in Figure 10, it is seen that the first external factor triggering mathematical perception and thinking is the quantity in the first column. As seen in the second column, the amount can take two different forms. Countable quantities are called discrete quantities, while continuous quantities are called magnitudes. The small (<5) or large (>5) discrete quantities also affect the action to be used for the enumeration. While small amounts can be detected at a glance without counting or estimation by means of parallel processing, estimation and calculation can be activated for large quantities. Perception of small quantities are exact and present in infants possibly at birth (Antell & Keating, 1983). Large quantities on the other hand is not exact and perceived approximately (Lipton & Spelke, 2003). There is an interaction between exact and approximate number system and practicing non symbolic approximate number leads to an improvement in exact arithmetic in school (Hyde, Khanum, & Spelke, 2014)

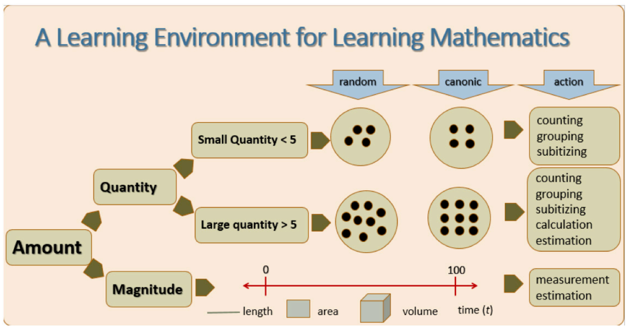
Canonically or randomly arranged discrete quantities also affects the action to be used for enumerating (Krajcsi, Szabo, & Morocz, 2013). Randomly arranged quantities, which make grouping and calculation relatively difficult, encourage the individual to guess if the given time is short, count if the time is sufficient. However, noticing the regularities in the canonically ordered quantities can create the opportunity to use the groupings and different calculation actions. We use the actions of measuring and estimating to determine the amount of things called magnitudes, such as length, area, volume, time etc. We can summarize the quantification process of mathematics as in Figure 10.

If we think of mathematization as a quantification process; we can say that the main triggering thing used in this process is the amount. The main action that governs this process is the determination of this amount. There are various sub-actions used to perform this main action. These actions may differ according to the type, shape and time of appearance of the quantity or magnitude. For example, estimation, measurement and calculation actions can be performed when determining a continuous quantity, while counting, grouping and calculation actions can be used to determine a discrete quantity. If the number of a canonically arranged and large "quantity" needs to be found in a very short time, the estimation action is triggered, while grouping and calculation can also come into play as the given time increases.

As a result, as can be seen, this context provides the framework for an important part of basic mathematics. It can be said that it will be possible to conduct mathematics education more effectively and efficiently in learning environments where the variables mentioned in this section can be controlled and manipulated. It is hoped that this framework, which is theoretically at the beginning and quite crude, will mature with additional research and theoretical studies.

Figure 10

A learning environment that triggers mathematical actions and thinking



When we consider the issue from the perspective of mathematics learning disability or dyscalculia, we see that individuals can perceive mathematical concepts or relationships at different levels and forms. As different representations activate different parts of the brain (Dehaene, Piazza, Pinel, & Cohen, 2003; Vogel, Goffin, & Ansari, 2015), the probability of learning the concept increases. For this reason, we can say that each mathematical concept to be taught will be more effective and productive with a teaching environment prepared in accordance with the triple coding theory or multiple representations of content (Sankey, Birch, & Gardiner, 2011). In fact, there are studies in this direction (Cohen Kadosh, Dowker, Heine, Kaufmann, & Kucian, 2013; Kucian et al., 2011; Ozdem & Olkun, 2019) in the current literature that show the efficacy of the basic mathematical skills training, such as subitizing and conceptual subitizing (Clements et al., 2019). It is seen that such intervention studies (Groffman, 2009; Olkun & Özdem, 2015; Ozdem & Olkun, 2019), which aim to develop different aspects of the basic number processing system as a whole, are more effective than traditional methods.

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The Effects of a Computer-Based Mathematics Intervention in Primary School Students with and Without Emotional and Behavioral Difficulties

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Abstract

Mathematics difficulties (MD) affect about 20% of the students in German schools. Almost half of them also exhibit emotional and behavioral difficulties (EBD). While a growing number of mathematics interventions target children with MD separately, there is a lack of evidence for the effectiveness of these interventions for children with combined MD and EBD. This study aims to investigate the differential effects of an evidence-based mathematics intervention on children with and without EBD.

This single-case study examined 11 children with internalizing and externalizing EBDs from grades 3 and 4 using a staggered AB-Design. A computer-based mathematics intervention was provided for 5 weeks, during which the mathematical performance of the students was measured using a learning progress assessment in A- and B-phases. Data were analyzed using (a) overlap indices, (b) piecewise linear regression (PLM) models for each student, and (c) a multilevel PLM across all children. The results suggest different effectiveness for children with and without EBD, indicating a small direct influence of the severity of the EBD. Thus, the effectiveness of mathematics interventions might not be generalizable for children with combined EBD and MD. Further research is necessary to better understand the differential effectiveness of mathematics interventions for these children.

Keywords:

Mathematics Difficulties; Single Case Study; Emotional and Behavioral Difficulties; Mathematics Intervention;

Introduction

Several studies have shown that about 20% of students in German schools have severe difficulties with learning mathematics (OECD, 2019; Frey et al., 2010). Typically, children with mathematics difficulties (MD) struggle with the basic operations (Kuhn, 2015), place value understanding (Gebhart et al., 2012; Moeller & Lambert, 2019), and number sense (Kuhn, 2015). This 20% value has remained stable across the recent twenty years in grades 3 to 9. It has to be stressed that the empirically found prevalence is substantially bigger

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than the expected prevalence based on the definition of developmental dyscalculia (DD) as defined in the ICD-11 (WHO, 2022).This means that there is a large achievement gap and that the actual percentage of students with low mathematical skills is likely to be higher than the ICD-11 definition allows (Ehlert et al., 2012; Schulte-Körne, 2021) which underlines the need for school-based, non-therapeutic interventions.

About half of the German students with MD also exhibit emotional and behavioral difficulties (EBD) in at least one domain (Visser et al., 2020). EBD can basically be differentiated into internalizing (e.g., depression) and externalizing (e.g., attention-deficit/hyperactivity disorder; ADHD) disorders (Achenbach & Edelbrock, 1978). In a synthesis of epidemiological studies conducted by Visser et al. (2020), about 30% of students with MD even exhibited EBD in more than one domain. Students with MD had an especially high vulnerability for ADHD (odds-ratio=3.7), depression (3.25), and anxiety disorder (2.26) compared to students without any learning difficulty. Against the background of the reported prevalence rates, one out of ten students in Germany shows comorbid MD and EBD. Given a typical German class with nearly 30 students, there are statistically about three students with comorbid MD and EBD in every class. Thus, one could conclude that EBDs are typical comorbid disorders for students with MD. Furthermore, math growth trajectories of students with emotional difficulties from ages 7 to 17 were shown to be significantly lower than those of students without comparable problems (Wei et al., 2013). Results from a large-scale study conducted in the US with over 9000 students from kindergarten through grade 8 also indicate that low performance in mathematics (even after statistically controlling for reading proficiency) significantly increases the risk for developing poor interpersonal skills and internalizing behavioral problems (Lin et al., 2013).

At least four hypotheses have been raised to explain the comorbidity of learning and behavior disorders (see Morgan & Sideridis, 2013). First, learning problems might cause behavioral problems because learning problems could lead to disengagement and more disruptive behavior in the classroom. Second, the behavioral problems could interfere with the demands of academic learning situations, such that students' problematic behaviors significantly affect their learning performance. Third, it would be possible that learning and behavioral problems are reciprocally or transactionally related, i.e., learning problems negatively affect behavior, but these behavioral problems in turn negatively affect academic learning. And fourth, the two phenomena may be unrelated and other individual, contextual, or cultural factors could cause the comorbidity of learning and behavioral problems. Regardless of which explanatory model is applied in a specific case, it is important to consider these influencing factors when evaluating and developing evidence-based interventions (Morgan & Sideridis, 2013). Although there are few studies that examine the true underlying causal effects, there is a slight tendency toward viewing behavioral problems in particular as causing learning problems (Kulkarni et al., 2020). For this reason, we focus on these causal hypotheses in this study and further elaborate on related findings for evidence-based practice of comorbid math and behavioral problems.

There is currently a growing number of evidence-based interventions that underpins their positive effect for children with MD (Chodura et al., 2015; Jitendra et al., 2021; Reynvoet et al., 2021; Stevens et al., 2018). Typically, mathematics interventions focus on basic mathematical competencies such as number sense (e.g., subitizing, number line estimation, magnitude comparison), basic operations, or word problems. However, the effectiveness of mathematics interventions in meta-analyses differs substantially. For instance, Chodura et al. (2015) report effect sizes ranging between -2.31 and 5.09, and Jitendra et al. (2018) found effect sizes between −0.92 and 3.04. This highlights the importance of considering differential effectiveness, e.g., for different groups of children with MD. For example, Stevens et al. (2018) found lower average effect sizes for students from secondary schools than Chodura et al. (2015) found for primary school students. This result indicates that younger students with MD may benefit more from mathematics interventions than older students. A possible explanation could be the similarity of the contents between the mathematics interventions and the primary school mathematics curricula.

In recent years, international research on the topic has focused on computer-based interventions for mathematics. With emerging technological possibilities and increasing accessibility even for less privileged children, computer-based interventions promise to play a more and more important role for mathematics intervention in the future (Räsänen et al., 2019). In general, computer-based interventions can successfully support students in learning mathematics (Higgins et al., 2018; Buyn & Juong, 2017; Ran et al., 2021; Räsänen et al., 2009). Focusing on computer-based educational games, Buyn and Juong (2017) found an overall effect size of d=.37 with a range of .01 to 3.17. In another study examining low-performing students in particular, Ran et al. (2021) reported a substantial overall effect size of d=.54 with a range of -1.63 to 2.24. Computer-based interventions were especially effective in primary school, whereas secondary school students benefited less from computer-based interventions. The particular effectiveness in primary school might be the result of the typical contents in computer-based interventions, which are number sense and basic operations (Räsänen et al., 2019).

Recounts of the advantages of computer-based interventions in contrast to traditional approaches often mention the motivational effect of computer-based interventions. However, the empirical basis of this claim is rather tentative, with lower effect sizes for motivation than for mathematics performance (Higgins et al., 2018; Wouters et al., 2013).

In view of the fact that half of the students with MD also have comorbid EBD, this group of students deserves more attention. Peltier et al. (2021) recently presented a meta-analysis of single case studies on mathematics interventions for students with EBD. Of the 19 studies included, the majority (13) had been published before the year 2000. This finding indicates that there is no increase in published intervention studies regarding children with comorbid MD and EBD. In contrast, Reynvoet et al. (2021) found an exponential increase in mathematics interventions studies in general beginning from 2010, showing that while the general interest in mathematics increased drastically, children with EBD did not receive adequate attention in such research.

Peltier et al. (2021) reported positive effects of mathematics interventions for children with comorbid MD and EBD, and investigated intervention, context, and depended variable factors that might influence the effectiveness of interventions. The overall effectiveness of mathematics interventions for students with EBD in terms of Tau-U was 74.4%, which means that nearly three out of four comparisons of measurement points in intervention and baseline phase were improved (Peltier et al., 2021). There were no differences in effectiveness regarding the participants' age and interventionist (e.g., teacher or researcher). No significant differences were found regarding the duration of the intervention. Interventions that were conducted in separate rooms in the schools were significantly more effective than interventions that were conducted in the classroom. Interventions targeting accuracy were significantly more effective than interventions targeting the productivity (number of completed tasks), while the targeted mathematical concept (e.g., fact retrieval) had no impact on the effectiveness.

Peltier et al. (2021) comprehensively reported effectiveness factors regarding context and intervention. Most of their findings are in line with older reviews on children with comorbid MD and EBD (Hodge et al., 2006; Ralson et al., 2013). However, there is a lack of research on the effectiveness of interventions for children with comorbid MD and EBD that focusses on the type and severity of the EBD. Both externalizing and internalizing EBDs can be associated with different typical symptoms, causes, and environmental interactions (e.g., Farmer et al., 2016; Landrum, 2017). Current shifts from a categorical to a dimensional perspective on EBD suggest that it would be advisable to not only include the type, but also the severity (Zimmermann et al., 2019), especially because the dynamic interrelation between both constructs might be causal (e.g., Hinshaw, 1992). For these reasons, the dynamic interactions between learning and behavioral problems should be considered and specifically addressed when evaluating intervention effects (Morgan & Sideridis, 2013). In the following sections, we will describe theory-driven and empirically underpinned explanatory models that explain how the type and severity of one internalizing (math anxiety (MA)) and one externalizing (attention deficit/hyperactivity disorder (ADHD)) EBD might affect mathematical learning. The examples of MA and ADHD were chosen, because there are detailed theories regarding their effects on mathematical learning. In addition, MA and ADHD have substantial comorbidities with MD (Du Paul et al., 2013; Orbach et al., 2019).

Internalizing EBD and Mathematics – Math Anxiety

MA is described as "the feeling of tension and anxiety that interferes with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (Richardson & Suinn, 1972, p. 551). As MA refers nearly exclusively to arithmetic, everyday situations such as paying the groceries or estimating the time for a bus ride can lead to symptoms that are typical for anxiety disorders. MA symptoms can cover, for example, sweating, nervousness, increased heart rates, and palpitation (perception of the person's own heart beat) (Haase et al., 2019).

MA is associated with lower performance in arithmetic, as shown by a growing body of studies (Namkung et al., 2019; Sorvo et al., 2017; Zhang et al., 2019). Studies report short- and long-term negative effects on mathematics learning outcomes. There are inconsistencies in the literature regarding the question, at what age the association between performance in mathematics and MA emerges. While some studies identified negative effects on mathematics performance in primary school students, other studies did not find a negative association until secondary school (for a more detailed discussion see Orbach et al., 2019). It must be noted that MA does not affect all students negatively. Students with high intelligence seem to be more susceptible to having their mathematics performance inhibited by MA (Ramirez et al., 2016). In addition, many studies report gender differences in MA, indicating that girls are more prone to MA (Haase et al., 2019).

By adapting a common construct from the field of psychotherapy, Orbach et al. (2019) differentiated between state-MA and trait-MA. While state-MA refers to the (more or less uncontrollable) mental and physical reactions in stressful mathematical situations, trait-MA refers to the general self-concept of being math-anxious. This differentiation could explain inconsistencies in MA assessment methods and corresponding prevalence (Orbach et al., 2019). In addition, state-MA and trait-MA might be involved in two different explanations for the negative effect of MA as an internalizing EBD on mathematical learning. Of course, trait-MA and state-MA – and thus the corresponding explanations – do not exclude each other, but can coincide.

First, MA can lead to avoiding situations that entail the need to use mathematics. In school, this mostly refers to mathematics classes. Students with MA that avoid mathematics classes or do not pay attention during class have less opportunities to gain and practice mathematical expertise (Ashcraft & Moore, 2009). As a consequence, students with MA have lower mathematics skills and fail more often in tests. A repeated experience of failure, potentially combined with falling short on teachers' or parents' expectations might even increase MA. As avoidance behavior starts even before entering a mathematical situation, trait-MA appears to be more likely to cause avoidance behavior than state-MA.

Second, stressful situations are likely to draw individuals' attention to the anxiety symptoms. While solving mathematical tasks, students with MA might focus more on their fear and negative thoughts than on processing the tasks. Thus, MA might block working memory resources and paralyze the thoughts of students (Suárez-Pellicioni et al., 2016). Because the working memory impairing effects of MA only occur in mathematical situations, state-MA is more likely to block working memory resources than trait-MA.

Externalizing EBD and Mathematics – ADHD

ADHD is a neurodevelopmental disorder that can be described by three symptoms – attention deficit, impulsivity, and hyperactivity – that affect children's behavior in every-day as well as learning contexts independently from specific situations. As a consequence, children with ADHD symptoms show lower school performance (Arnold et al., 2020). One main cause for ADHD symptoms are lower executive functioning (EF) resources (Willcutt et al., 2005). Executive functions (especially inhibitory control, working memory, and cognitive flexibility) play an important role in school, as they are particularly challenged in numerous demanding situations (e.g., individual periods of quiet work, organization of the work process, examination situations). Among others, EFs are necessary for monitoring complex tasks as well as storing or retrieving information in or from the short-term memory (working memory), flexibly switching between different tasks (shifting), and inhibiting interfering stimuli (inhibition) (Gilmore & Cragg, 2018). EF play a crucial role in children's academic, emotional, and social development; inhibition skills, in particular, are relevant for self-regulation (e.g., in conflict situations) (Bailey & Jones, 2019; Cantor et al., 2019). However, especially in students with ADHD commonly have impaired executive functions (e.g., Barkley, 2015; Pineda-Alhucema et al., 2018). In a conceptual model of the relationship between EF and ADHD, Barkley (1997) shows that problems in inhibition affect working memory, emotional self-regulation, and cognitive flexibility, which in turn can lead to difficulties in behavioral self-regulation and thus ADHD symptom-specific behaviors.

Compared to their non-impaired peers, children with ADHD show lower mathematics performance, especially regarding fact retrieval and calculation (Orbach et al., 2020; Tosto et al., 2015). Against the background of different ADHD subtypes, attention difficulties affect mathematical performance stronger than impulsivity or hyperactivity (Massetti et al., 2008; Tosto et al., 2015). Overall, there are only a few high-quality studies examining the causal relationships between externalizing behavioral problems and learning problems; however, evidence tends to indicate that early externalizing behavior problems causally influences later academic performance (Kulkarni, Sullivan & Kim, 2020). In this context, hyperactive-impulsive behaviors in particular appear to have stronger predictive validity than oppositional-disruptive behaviors (Hand & Lonigan, 2021). Correspondingly, two explanations for lower mathematical performance in children with different ADHD profiles can be postulated.

First, children with attention deficits are prone to missing important information taught in school. Usually, lessons in schools last at least for 45 minutes, which might be longer than some children with ADHD can maintain attention. Over the course of several years in school, the probability of missing important information cumulates and leads to growing delays in mathematical development. This explanation is bolstered by studies showing that inattention is stronger related with low mathematical (and generally academic) performance than other ADHD subtypes (Massetti et al., 2008; Tosto et al., 2015).

Second, some – but not all (Willcutt et al., 2005) – children with ADHD also have low EF resources. The relevance of EF for mathematical performance has been demonstrated in several studies (see Friso-van den Bos et al. (2013) and Peng et al. (2016) for reviews). All three main components of EF are relevant in mathematical contexts: Working memory is particularly involved in retrieving arithmetic facts and monitoring complex calculations; shifting is necessary when different operations are embedded in one task; inhibition is relevant for suppressing solutions to similar tasks (Gilmore & Cragg, 2018).

The given examples for the implications of internalizing and externalizing EBDs for mathematics underpin that the same phenomenon –benefiting less from instruction or cognitive impairments while performing calculations – may be caused by different types of EBDs. As a consequence, internalizing and externalizing EBDs can amplify each other in similar phenomena.

Research Questions

As described above, MD in general and also the comorbidities with EBD in students are very common and particularly challenging in school practice. The development and intercorrelations of the two phenomena are complex, with EBD possibly even causing the development of MD. These mechanisms must be considered when designing and evaluating interventions that can be adaptive and appropriately targeted. So far, however, there are only few studies that address this challenge. Furthermore, the relevance of the type and severity of EBD for mathematical learning raises questions regarding the impact of internalizing and externalizing EBDs on mathematics interventions. Therefore, the current study will investigate the following four research questions:

1. How do mathematical skills develop in students with and without behavioral difficulties in the course of an evidence-based computer-based mathematics intervention?
2. With what pattern (i.e., immediately or continuously after implementation) do potential intervention effects set in?
3. To what extent do developmental trajectories of math skills during intervention differ between students with externalizing, internalizing, and no behavioral disorders?
4. To what extent do the severity of the externalizing and internalizing EBD influence the mathematical skills development during the intervention?

Method

Sample

A total of *N* = 11 students from 3 German primary schools participated in this study. Written consent was obtained from the parents in advance. 5 students were in grade 3 and 6 students were in grade 4. With three exceptions (2 Albanian, 1 Polish), all students spoke German as their first-language. All children showed low performance in a standardized math test (T-score ≤ 43 for all children; T-score < 40 for 8 students). Based on their EBD profile, the students can be categorized as having no or few difficulties (EBD-N, *n* = 4), predominantly internalizing difficulties (EBD-I, *n* = 5), or predominantly externalizing difficulties including attentional difficulties (EBD-E, *n* = 2). Table 1 provides an overview of the participating students including their mathematical performance and EBD profiles.

Instructors

The intervention was conducted by three female university graduates at the end of their bachelor studies. All of them had recently completed a course on intervention strategies for students with learning

Table 1

Overview of the participants

Pseudonym	Age (years; months)	Gender	L1	Grade	Math (T-score)[1]	CBCL	CBCL-Int	CBCL-Ext
EBD-N								
Clara	10;5	female	GER	3	32	12	0	6
Dana	12;0	female	GER	4	41	15	2	1
Emma	11;5	female	GER	4	43	18	3	1
John	11;6	male	GER	3	35	8	0	2
EBD-I								
Aron	10;1	male	GER	4	26	69	21	7
Ben	10;7	male	GER	4	35	47	17	2
Gloria	9;4	female	GER	3	30	37	14	0
Hugo	9;9	male	GER	3	30	39	14	0
Keanu	10;2	male	ALB	3	33	40	14	5
EBD-E								
Fabia	11;10	female	ALB	4	43	39	1	27
Ines	9;1	female	POL	3	39	44	8	13

Note. L1=First language; 1Math scores obtained from HRT 1-4 (Haffner et al., 2005); CBCL= scores from the CBCL (Döpfner et al., 2015); CBCL-Int=subscale "internalizing problems" of the CBCL; CBCL-Ext=subscale "externalizing problems" of the CBCL.

difficulties in mathematics and had also gained teaching experience in internships. They also received additional training on the implementation of all instruments and the intervention itself from scientific staff.

Design

A controlled single-case research design was applied for four reasons: First, this methodological approach allows us to examine the response to an intervention of individual students or smaller groups of students with shared characteristics (Riley-Tilman et al., 2020). Second, single-case research allows to capture important characteristics of individual students that might explain intervention success (Riley-Tilman et al., 2020). Third, the repeated and close-meshed measurements in a baseline and intervention phase allow for a systematic comparison of developmental trajectories without and with intervention, as well as specific patterns of intervention effects, which in turn can be used to develop evidence-based support methods (Huitema & McKean, 2000). Fourth, the approach is highly feasible, especially for studies with small target populations (such as students with special education needs) (Maggin et al., 2018).

This study used a quasi-experimental controlled single case AB-design with multiple baselines across participants. Specifically, this means that several students with different characteristics (in our case, different forms of EBD) participated in the study so that single case trajectories can be compared between baseline rates. The intervention onset was staggered across the individual cases so that potential intervention effects were more likely to be ascribed to the implementation of the intervention. The length of the time series measurement for the baseline phase varied between three and five measurement points across all students. The length of the time series measurement for the intervention phase varied between three and nine measurement points across all students. The different lengths of the A- and B-phases were caused by the school closures during the pandemic, which affected the initial design plans.

Instruments

Math performance: Math performance was assessed with the Heidelberger Rechentest 1-4 (HRT) (Haffner et al., 2005), a standardized math test for German primary school students. The timed test covers the basic operations (addition, subtraction, multiplication, division), complement tasks, and number comparison. The retest reliability of the HRT is sufficient (r_{tt}= .77-.89)

Students' development in math performance was measured by a computer-based progress monitoring Cody-LM (Schwenk et al., 2017) embedded in the intervention program. The Cody-LM covers addition, subtraction, and number ordering tasks in a timed condition. Depending on the reaction time, students gain virtual coins for their correct answers. However, when the students' answers are wrong, coins are withdrawn correspondingly. The psychometric validity of the Cody-LM has been tested empirically and showed good split-half reliability ($r_{split\text{-}half}$ = .87-.93) (Schwenk et al., 2017).

Behavior: Students' emotional and behavioral problems were assessed with the German version of the Child Behavior Checklist – Teacher Report Form (CBCL-TRF) (Döpfner et al., 2015). The CBCL-TRF covers internalizing (anxiety, depression, withdrawal, and somatic complaints), externalizing (breaking rules, aggressive behavior), and attentional problems (inattention, hyperactivity-impulsivity). Students' behavior was assessed using a 3-point Likert scale completed by the classroom teachers. Studies examining the psychometric properties of the German version of the CBCL-TRF suggest good internal consistencies for Externalizing Problems (α = .94 - .96), Internalizing Problems (α = .87), and Attention Problems (α = .93 - .94) (Döpfner et al., 2011; Volpe et al., 2018).

Intervention

Students were trained with the computer-based mathematics intervention Meister Cody (Kasaa Health, 2013). Meister Cody is based on a robust indicators approach: Skills that predict mathematical learning well are trained to provide students with a sound basis for subsequent learning. The robust indicators cover number line estimation, transcoding, fact retrieval, part-whole-tasks, number-set-correspondence, calculations, word problems, and working memory tasks (Kuhn & Holling, 2014). Example screenshots of the intervention formats are shown in Figure 1. After an initial assessment, the training content is individually adapted to the students' mathematical profile. The effectiveness of Meister Cody has been tested in an empirical study (Kuhn & Holling, 2014).

The computer-based intervention was conducted by the instructors on a tablet in a quiet and separate room in school. Training sessions lasted for about 20 minutes each. Due to difficulties in the implementation of the study caused by the COVID-19 pandemic, students only received between 3 and 9 training sessions.

Results

The analysis of the data obtained in the current study was structured into three sections. First, the trajectories of students' mathematical performance were visually analyzed, including descriptive analyses and overlap indices. Second, piecewise regression models were employed for each student individually to test for significant intervention effects. Third, a hierarchical piecewise regression aggregating all students and

Figure 1

Screenshots of the intervention program. Top left: magnitude comparison; Top right: number line estimation; Bottom left: counting; Bottom right: addition.

EBD profile were used to investigate the influence of EBD type and severity. All analyses were conducted using R (R Core Team, 2018) and the package scan (Wilbert & Lüke, 2021).

Visual Analyses and Overlap Indices

Based on the visual analysis and the overlap indices, the intervention had different effects on the different students: While some students benefited well from the training, others showed stagnating or even decreasing performance trajectories. To examine the intervention, we calculated several non-overlap measures. The non-rescaled non-overlap of all pairs (NAP; Parker & Vannest, 2009) is the percentage of all pairwise comparisons across the baseline and intervention phases. According to Parker and Vannest (2009), medium effects are indicated by values of 66% to 92%, and strong effects are indicated by values of 93% to 100%.

The Percentage of All Non-Overlapping Data (PAND) indicates the percentage of data from the baseline and intervention phases that do not overlap. There are no conventions for interpreting the PAND value, but there are certain rules of thumb. For example, a PAND value of 50% or less indicates that the differences between the baseline and intervention phases occurred by chance. A value of 70% or more could indicate a small effect, 80% or more a medium effect, and 90% or more a large effect (Parker et al., 2007).

Percentage Exceeding the Median (PEM; Ma 2006) indicates the percentage of data points from the intervention phase that are above the median of the baseline phase. The PEM can take values between 0 and 100%, with values between 70% and 90% indicating a moderate effect and 90% and above indicating a strong effect (Alresheed et al., 2013).

The Percentage Exceeding the Trend (PET) indicates the percentage of data points from the intervention phase that are above the trend from the baseline phase. It is therefore the trend-based equivalent of the PEM. The PET can take values between 0 and 100%, with values between 70% and 90% indicating a moderate effect and 90% and above indicating a strong effect (Alresheed et al., 2013).

Tau-U analysis allows to examine treatment effects on both between-phase difference and within-phase trend (Parker et al., 2011), and offers at least four different types of Tau-U calculations (Parker et al., 2011). In this study, the Tau-U "non-overlap with phase B trend with baseline trend controlled" (Parker et al., 2011, p. 291) was employed, which is the non-overlap of all pairs between the baseline and intervention phase plus the intervention phase trend minus the baseline phase trend. Although no general recommendation can be made about conventions for interpreting Tau-U values, a value of .20 can be considered as a small change, values from .20 to .60 as moderate changes, values from .60 to .80 as large changes, and values above .80 as large to very large changes (Vannest & Ninci, 2015).

Comparing the trajectories of the three EBD groups, all students with no EBD benefited at least slightly from the intervention (mean Tau-U=.35). In contrast, about half of the students in the EBD-I (mean Tau-U=.16) and EBD-E (mean Tau-U=-.09) groups did not benefit from the intervention at all. However, especially two out of five students from the EBD-I group showed considerable responsivity (Tau-U=.44 and .47, $p < .05$ in both cases). Descriptive statistics and overlap indices of all students are summarized in Table 2.

Besides these differential (average) effects for students with and without EBD, the effects of the intervention are generally low. The training effects are significant in only two cases. The visual analysis of the learning trajectories indicates mostly stable performances in the A- as well as in the B-phases. Moreover, level-related overlap indices such as Percentage Exceeding the Mean (PEM), Non-overlap of All Pairs (NAP), or Percentage of All Non-overlap Data (PAND) are substantially higher than trend-related indices such as Percentage Exceeding the Trend (PET). These findings indicate that predominantly level effects can be assumed, but barely any trend effects: Students might perform better during the intervention, but their development is – at least across the observed time – not accelerated. No to little trend effects of the intervention in all students might also be caused by positive trends in the baseline phase in nine out of eleven cases.

As there are little to no trend effects in the observed B-phases, there is no evidence for continuous intervention effect across time in this study. In contrast, those students who benefited from the intervention immediately showed increased mathematics performance. Only in the EBD-N group did all students show small positive trends in the B phase, which indicates that these students may have also benefited continuously to a small degree.

When investigating the variance in the progression monitoring, students with externalizing EBD in particular showed great variance across time. Compared to students with externalizing EBD, variance was generally lower in the EBD-N and EBD-I group.

Piecewise Regression Models

To investigate the level and slope effects of the intervention in the three EBD groups in more detail, piecewise regression models were run separately for the different groups. Piecewise regression models can bolster assumptions of (a) level effects in terms of an immediate effect of the intervention on mathematical performance and (b) continuous increase in performance over a longer time period. As the visual analysis and the overlap indices indicated that there are little to no trend effects in the B phase, trend effects were excluded from the regression models.

All groups showed higher level parameters than slope parameters. In addition, the regression parameters both for levels and for slopes were bigger in the EBD-I group than in the EBD-E group, and even bigger in the EBD-N group. Especially regarding slopes, regression parameters were close to zero. These findings underpin the results of the visual analyses, which indicated an immediate effect that was strongest in the EBD-N group and weakest in the EBD-E group.

The piecewise regression models employed in this analysis yielded no level or slope effects that were statistically significant. Given the comparably small number of measurement points, we argue that the regression parameters may be of value for practical decision-making in interventions planning, although the hypothesis concerning significant level and slope effects must be rejected. The explained variance supports the notion of small level effects and negligible slope effects in all groups.

Hierarchical Piecewise Regression Models

Finally, a multilevel extension across all cases was calculated with measurements at level 1 nested in subjects at level 2 (Van den Noortgate & Onghena, 2003). In addition to the standard regression model, two interaction effects between intensity of EBD and level and slope were included. This application of regression models allows for inferences about the intervention effects across all students considering the influence of the severity of the specific EBD on the math competence trajectories in the B phase. The overall explained variance of the hierarchical piecewise regression model was R^2=.597. The parameters of the hierarchical piecewise regression model are summarized in Table 4.

The results indicate a significant level effect across all students, meaning that there was an improvement in math skills immediately after the implementation of the intervention. This result is in line with the findings from the previous analyses. None of the other variables had a statistically significant effect. Non-significant effects might be caused by the relatively small number of measurement time points, especially in the B phase (see Table 4), which means that more attention must be paid to the regression coefficients, as these indicate the practical significance of the competence development during the intervention phase. Severity of internalizing EBD was associated with lower intervention effects to a small degree, whereas severity of externalizing – including attentional difficulties – had no considerable effect. In addition, the model shows no substantial interaction effect of level and intensity of EBD.

Figure 2

Trajectories of the mathematical performance of the EBD-N group.

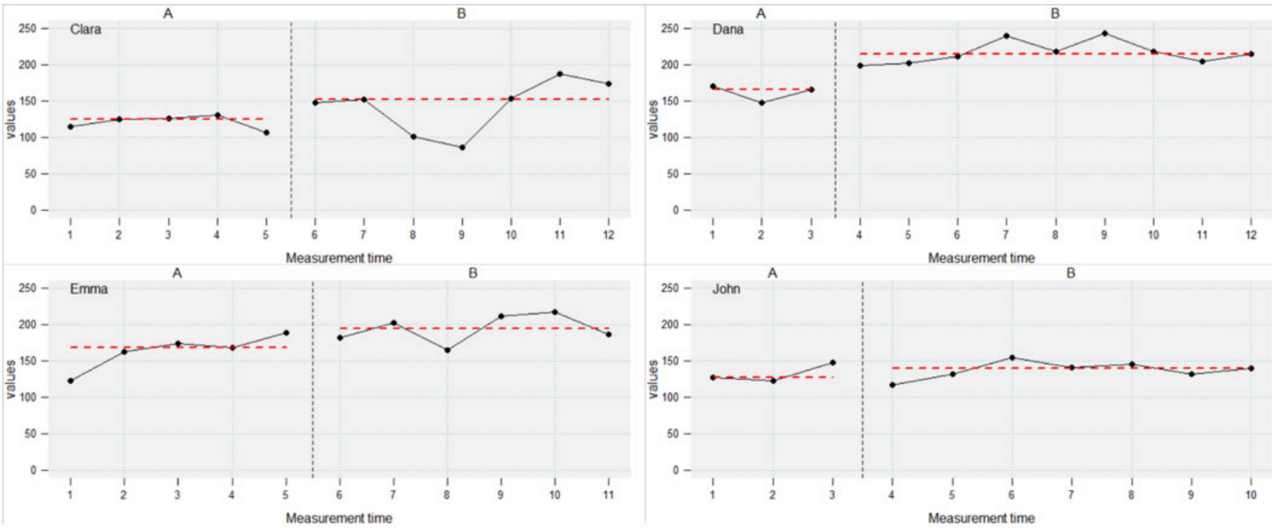

Figure 3

Trajectories of the mathematical performance of the EBD-I group.

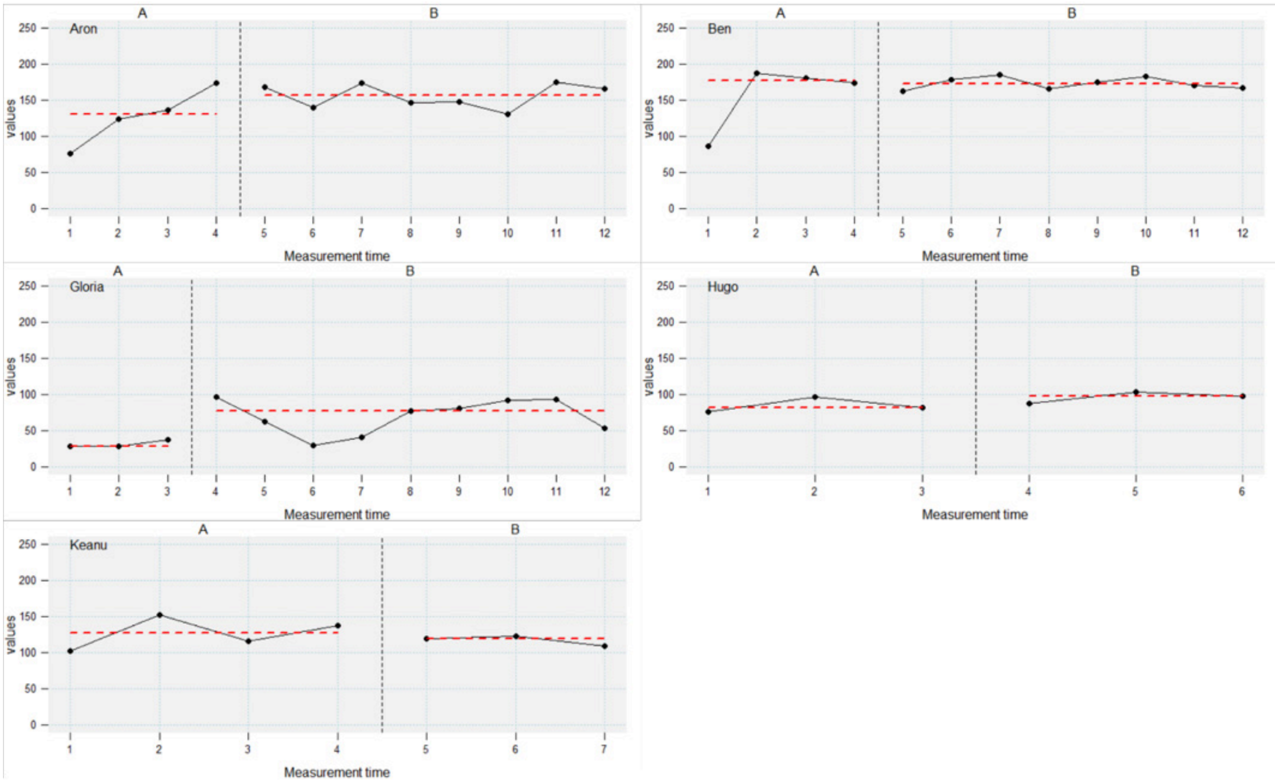

Figure 4

Trajectories of the mathematical performance of the EBD-E group.

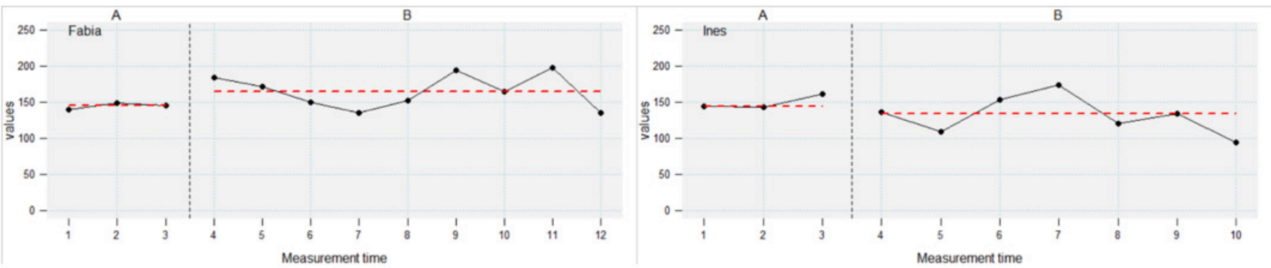

Table 2

Descriptive statistics of the A- and B-phases

Pseudonym	A-Phase					B-Phase					Overlap					
	MP	*M*	*SD*	Md	Trend	MP	*M*	*SD*	Md	Trend	PND	PEM	PET	NAP	PAND	Tau-U
EBD-N																
Clara	5	120.80	9.65	125	-1.00	7	143.00	26.78	152	7.14	71.4	71.4	71.4	71.4	66.7	.33
Dana	3	161.33	11.72	166	-2.00	9	216.44	15.50	214	1.40	100	100**	100**	100**	100**	.59**
Emma	5	162.60	24.65	168	13.80	6	193.50	20.07	194	3.34	50.0	83.3	.00	83.3*	81.8**	.31
John	3	132.67	13.43	127	10.50	7	137.14	11.71	139	2.04	14.3	85.7	.00	61.9	80.0	.18
EBD-I																
Aron	4	127.25	40.03	130	30.30	8	130.00	16.78	157	.50	12.5	100*	.00	76.6	83.3*	.20
Ben	4	156.75	47.46	177	25.70	8	173.25	8.35	172.5	-.02	.00	37.5	.00	43.8	50.0	-.06
Gloria	3	31.33	5.77	28	5.00	9	69.56	24.25	77	1.35	88.9	100**	66.7	96.3*	83.3	.44*
Hugo	3	84.67	10.26	82	3.00	3	96.00	8.19	98	5.50	66.7	100	66.7	88.9	66.7	.47
Keanu	4	126.75	22.14	126.5	6.90	3	117.00	7.21	119	-5.00	.00	.00	.00	41.7	42.9	-.24
EBD-E																
Fabia	3	144.33	5.03	145	3.00	9	164.67	23.60	164	-.52	77.8	77.8	44.4	77.8	66.7	.20
Ines	3	149.33	10.12	144	8.50	7	131.29	26.63	124	-3.89	14.3	28.6	.00	23.8	40.0	-.38

**Note. MP = measurement points; M = mean; SD = standard deviation; Md=median; PND = percentage of nonoverlap data; PET = percentage exceeding the mean; PET = percentage exceeding the trend; NAP = non-overlap of all pairs; PAND=percentage of all non-overlapping data; Tau-U = baseline corrected Kendall-Tau (Tarlow, 2017); *=p < .05; **= p <.01.*

Table 3

Piecewise regression parameters for the three EBD groups.

Parameter	B	SE	t	*p*	ΔR^2
EBD-N					.438
Intercept	144.468	15.192	9.510	<.01	.300
Level	14.338	10.153	1.412	.116	.134
Slope	3.428	1.841	1.862	.07	.004
EBD-I					.464
Intercept	106.040	18.539	5.720	<.01	.355
Level	12.552	10.063	1.247	.219	.105
Slope	1.668	1.829	.912	.367	.003
EBD-E					.454
Intercept	146.833	12.319	11.919	<.01	.158
Level	5.655	16.659	.339	.738	.289
Slope	-.738	2.594	-.284	.779	.007

**Note. B=Unstandardized regression coefficient; SE=standard error; t=value of the t-test; p=significance of t-test; ΔR^2=change in explained variance.*

Table 4.

Hierarchical piecewise regression model across all students.

Parameter	*B*	*SE*	*t*	*p*	ΔR^2
Intercept	109.707	25.077	4.375	<.01	.495
Level	23.969	11.121	2.155	.034	.098
Slope	1.616	1.176	1.374	.172	.001
Internalizing	-1.721	1.534	-1.122	.294	.002
Externalizing	.340	.760	.447	.667	.000
Level x Internalizing	-.265	.614	-.432	.667	.000
Level x Externalizing	-.380	.314	-1.209	.230	.000

**Note. B=Unstandardized regression coefficient; SE=standard error; t=value of the t-test; p=significance of t-test; ΔR^2=change in explained variance.*

Discussion

The aim of this study was to investigate the effects of an evidence-based mathematics intervention in students with and without EBDs. Special attention was given to the onset and development of potential intervention effects, the comparison of effects in students with and without EBD, and the potential influence of externalizing and internalizing EBDs on the intervention effects. In sum, four research questions were examined.

Regarding the first research question addressing students' mathematical development during an intervention, the study found substantial differences in responsivity. While a few students showed significantly better mathematics performance during the B-phase, other students' performance was similar or even lower than in the baseline phase. This finding underscores the importance of examining the individual effectiveness on a single-case basis for interventions that have shown to be effective in randomized control group studies (Riley-Tilman e al., 2020).

The second research question focused on patterns of intervention effects (i.e., immediate or continuous effects after implementation). The effects in those cases that showed positive intervention effects set in closely after the beginning of the B phase, as illustrated in the visual analysis. This interpretation is supported by overlap indices and piecewise regression model parameters that indicate a level effect. However, visual analyses, overlap indices, and piecewise regression models show no slope effects. Where found, performance improved immediately after the beginning of the intervention, but did not accelerate in the course of the intervention.

Third, we examined differences in math development trajectories between students with externalizing, internalizing, and without behavioral problems. Based on the results in the CBCL, students were assigned to three groups. In all analyses, the students without EBD (EBD-N) benefited the most from the intervention. Students with internalizing EBDs (EBD-I) benefited less, while students with externalizing EBDs (EBD-E) showed the lowest intervention effects. The pattern of effects in the three groups was the same for level and slope effects. These findings indicate a differential effectiveness of mathematics interventions in students with and without EBD, with lower effects for externalizing EBDs. Previous studies showed that students with EBD were likely to show lower mean math performance (Graefen et al., 2015; Wei et al., 2013), which is supported by these results. The results also indicate that students with externalizing EBDs such as ADHD are more strongly impaired than students with internalizing EBDs. As pointed out, externalizing EBDs are often associated with low executive functioning resources, which play a crucial role in mathematical learning. Thus, lacking executive capacity can explain these findings.

Fourth, we examined the extent to which the severity of externalizing and internalizing behavioral problems influenced the development of math skills during the intervention. The hierarchical piecewise regression model was employed to test the direct influence of the severity of students' EBD on the intervention effects. Although not significantly, the severity of internalizing EBD had a substantial negative effect on the performance in the B-phase, meaning that lower internalizing behavior problems are associated with stronger gains in math competence. A typical mathematics related internalizing EBD is math anxiety. The results suggest that the severity of anxiety (as one example for an internalizing EBD) has a direct effect on the students' responsivity to a mathematics intervention. A potential explanation could be that students with internalizing EBDs showed more avoidance behavior, even in the training sessions. No comparable effects were found for externalizing EBDs, nor the interaction of severity of internalizing or externalizing EBDs with level. This finding is in line with previous studies that have shown that students with MD are especially vulnerable to internalizing EBDs, such as anxiety or depression (Visser et al., 2020). With respect to practical settings, this could imply the need for modifications to intervention for students with internalizing behavior problems.

In general, the results of the current study should be interpreted with caution due to its limited external and internal validity: First, the size of the effects of the intervention might be limited due to few measurement points and training sessions in a short intervention phase. In particular the cases that showed no significant effects might have just needed more time. Whereas no slope effects were found in a short intervention phase, a longer intervention phase might reveal an acceleration in students' development. Second, the design lacked a withdrawal phase. In a withdrawal design, a second A-phase is added immediately after the B-phase, potentially followed by a subsequent second B-phase. An additional A-phase (and B-phase) would allow for disentangling the intervention effects from random or schooling effects. Thus, the significance of the results would be higher in a withdrawal design.

The investigation of the effect of a computer-based mathematics intervention on students with and without EBD employed a single-case design. Such a design was considered appropriate especially in view of the clearly outlined and specific target sample that does not expect high sample sizes, i.e., students with comorbid MD and EBD. However, it must be noted that results from a single-case study are hard to generalize for the whole population due to low sample sizes (Maggin et al., 2018), though replicating single-case studies does enable generalizability of findings from such studies.

Practical implications

The (tentative) results of the current study provide partial support for the assumption that effectiveness of mathematics interventions cannot be generalized for students with EBDs. Theory-driven explanatory models and empirical findings suggest that students with EBDs benefit less from mathematics interventions. Therefore, specific evidence for the effectiveness of mathematics intervention for students with EBDs as well as insights into the underlying theoretical mechanisms of effectiveness is necessary. The need for evidence-based mathematics interventions for students with EBDs is underpinned by the fact that about half of the students with MD are affected by EBDs, too.

Since externalizing and internalizing EBDs showed different effects on the intervention in this study, the differentiation into externalizing and internalizing EBD seems to be adequate. However, the findings regarding the effects in this study were inconsistent: While students from the EBD-E group showed practically no increase in performance in the visual analyses, overlap indices, and the piecewise regression models, the severity of the internalizing EBDs had a substantial direct influence in the hierarchical piecewise regression models. One reason might be that the students from the EBD-E group also had internalizing EBDs to some (lower) extent and vice versa. This explanation raises the question, how externalizing and internalizing EBDs interact and might amplify each other in students with both EBD subtypes. Future research on mathematics interventions for students with EBD might address this question.

Should future studies find evidence for the assumption of differential effects of mathematics interventions for students with EBD, there would be a need for specific interventions for these students. Based on the results of this study, EBD-sensitive mathematics interventions might focus on internalizing, externalizing, or combined EBDs. The explanatory models presented above suggest such a differentiation. In addition, a thorough review of effect models on mathematical learning for other EBDs that are less researched, such as depression, social problems, or oppositional behavior, might inform specific mathematics interventions for students with EBD.

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Examination of Home-Based Number and Operation Training Program on Early Mathematics Ability and Mother-Child Relationship*

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Abstract

This study aims to examine the effect of the "Number and Operation Training Program" on early mathematics ability and mother-child relationship at home. In an experimental design, pre-test, post-test, permanence test were used with control groups. A total of 21 children and their mothers participated in the research; consisting of 13 children in the experimental group and 8 children in the control group in the Aksaray province (in Turkey). Personal Information Form, Early Mathematics Ability Test-3, Child-Parent Relationship Scale and Mother Interview Form were used as data collection tools. The training program had been implemented for 13 weeks prior to the data collection. The study found that the training program increased children's scores on early mathematics ability and the positive relationship with their mothers. Mothers stated that the most important contributions of the training program to their children were "recognizing numbers, learning numbers, learning geometric shapes, knowing the total number of objects displayed in the group, learning new games and playing games with family members". They also stated that the most important gains for them were "better relationship and quality time with their children and increased use of mathematics in daily life".

Keywords:

Early Mathematics Education, Home Visit, Home Math Environment, Mother-Child Relationship

Introduction

The basic mathematical skills that children acquire at an early age have a significant impact on the development of mathematics skills and other academic abilities in the following years (Aunio & Niemivirta, 2010; Claessens et al., 2009; Huntsinger et al., 2016; Merz et al., 2014). Low counting skills have a negative impact on employment prospects and the economic status of countries (Kadosh et al., 2013). Given the impact of early academic skills on children's future academic success, employment prospects, and the economic status of countries; the importance of developing these skills of young children is evident.

Studies highlight some of the main factors that affect children's academic achievement and mathematical performance, which include the socioeconomic status of the family (Jordan & Levine, 2009), child's gender (Dickhäuser & Meyer, 2006), race and ethnicity (Cross et al., 2009). In recent years, there has been an increase in studies investigating the effects of the home math environment on children's mathematical skills (Daucourt et al., 2021; DeFlorio & Beliakoff, 2015; Hart et al., 2016; Kluczniok et al., 2013; Kwing- Cheung & McBride, 2017; Lombardi & Dearing, 2020; Niklas al., 2016; Niklas & Schneider, 2014; Sonnenschein et al., 2012; Susperreguy et al., 2020; Susperreguy & Davis-Kean, 2016; Zhang et al., 2019). These studies show that the quality of the home environment in which young children live, their parents' academic expectations, beliefs and attitudes towards mathematics significantly affect children's mathematical skills (Anders et al., 2012; Kleemans et al., 2012; Skwarchuk et al., 2014; Sonnenschein et al., 2012).

The Home Math Environment (HME), which includes all math-related activities, parental attitudes and expectations, resources provided to the child at home, and parent-child interactions, is considered a potentially promising environment in promoting children's early math development (Daucourt et al., 2021; DREME, 2020). The home math environment covers experiences in which children participate interactively with their parents as well as counting, recognizing numbers and playing logical games (LeFevre et al., 2009; Napoli & Purpura, 2018).

According to the "home mathematical skill model" proposed by Skwarchuk, Sowinski and LeFevre (2014), parental attitudes have an indirect effect on children's mathematical skills through their behaviours. This model emphasizes the role of the parent as the main determinant of the home math environment. Parents play an important role in early learning at home by providing a learning environment that will allow children to become aware of everyday situations (Cross et al., 2009; Hart et al., 2016).

Formal math activities at home denote experiences in which parents directly and purposefully teach numbers, quantity or arithmetic to their children to improve their knowledge of mathematics. In contrast, informal math activities refer to experiences or situations, such as cooking, measurement activities or crafts required in carpentry, quantity comparisons, spatial processing, which do not have a direct mathematical teaching purpose (Skwarchuk, Sowinski & LeFevre, 2014). Doing math activities appropriate to the developmental characteristics of children in the home environment is positively associated with children's early mathematics knowledge in kindergarten and primary school (Blevens-Knabe & Musun-Miller, 2016; Eason, & Ramani, 2018; Huntsinger et al., 2016; Levine et al., 2019; Sonnenschein, Metzger &, Thompson, 2016; Susperreguy & Davis-Kean, 2016; Thompson et al., 2017; Zippert & Rittle-Johnson, 2020). Studies show that mathematical activities based on the interaction between parents and children in the home environment improve children's mathematical skills and increase mothers' awareness of mathematics (DeFlorio & Beliakoff, 2015; Hojnoski et al., 2014; Kwing-Cheung & McBride, 2017; LeFevre et al., 2010; Manolitsis et al., 2013; McCarthy et al., 2012; Melhuish et al., 2008; Niklas et al., 2016). The most effective factor in the success of the mathematics intervention programs is the determination of the strong and predictive factors that affect mathematics achievement (Güleç & İvrendi, 2017).

Considering the results of the research, it is expected that examining the variables of "mother" and "home environment" in order to improve children's mathematical skills in this study will be effective in gaining early math skills. There are few studies on home-centered mathematics implementations in early childhood in Turkey (Güleç & İvrendi, 2017; Gürgah-Oğul & Aktaş-Arnas, 2020; Orçan-Kaçan et al., 2016; Uslu-Çavdarcı, 2016; Uzun, 2013). Therefore, this study aims to increase children's early mathematical abilities and improve mother-child relationships by implementing a mathematics training program at home.

The Current Study

The aim of the study is to examine the effect of the Number and Operation Training Program on children's early mathematics ability and mother-child relationship. The program is directed towards young children aged between 48-65 months old, who do not attend pre-school education institutions. The implementation of the program requires home visits by the researchers and the active participation of mothers.

Ethical Consideration

Mothers were informed about the data collection process, their rights and the rights of their children as participants, and the measures taken to ensure their confidentiality. The first author highlighted that no personally identifying information would be used in the study in order to protect the participants' anonymity. In addition, mothers were told that they could leave the study at any time, if they or their children did no longer want to participate.

Method

This study adopts a mixed-methods research design, which utilizes both quantitative methods to provide a more general understanding of the problem as well as qualitative methods to generate an in-depth awareness of the issue under investigation. Quantitative and qualitative data were used to examine the effect of the training program on the early mathematics ability of 48-65 month-old children and mother-child relationship. A hybrid method is a preferred method in research where a single data source is insufficient, a second method is required to develop the first method, and a general research goal can best be addressed in multiple stages or projects (Creswell & Plano Clark, 2011). In the quantitative part of the research, a semi-experimental pattern was used in multi-subject patterns from experimental research patterns. Experimental studies aim to test the effect of differences created by the researchers on the dependent variable. In experimental patterns, the main goal is to test the cause and effect relationship between variables (Büyüköztürk et al., 2014).

Participants

The participants of the study consists of 21 mothers and their children who do not attend any pre-school education institutions in Aksaray (a city of Turkey) and show normal development of 48-65 months. There are 8 children in the control group and 13 children in the experimental group. At the beginning of the study, the control group consisted of 12 mothers, however, 4 mothers with their children decided to leave the study due to health problems. Nine children in the experimental group were 48-59 months old and 4 were 60-65 months old. In addition, 5 of the children were girls and 8 were boys. Mothers of 8 of these children were aged between 20-35 years, and the rest of the mothers were 36 and above. Fathers of 7 of the children were aged between 25-35 years and the rest of the fathers were 36 and above. It was found that mothers of 9 of the children in the experimental group had education below high school level and 4 of them had high school or higher education; fathers of 8 children had education below high school level and fathers of 5 children had high school or higher education. It was determined that 2 of the children in the experimental group were the only child of their family, 8 of the children had two siblings, and 3 children had three or more siblings. It was found that 7 of the children in the control group were 48-59 months old and 1 was 60-75 months old. Four of these children were girls and 4 were boys. It was determined that mothers of 3 of the children in the control group were aged between 20-35 years and mothers of 5 of them were 36 years old and above. In addition, it was determined that fathers of 3 of the children were aged between 25-35 years old, and fathers of five of them were 36 and above. It was determined that mothers of 2 of the children in the control group had education below high school level, and 6 of them had a high school or higher education. Fathers of 4 of the children had education below high school level, while fathers of 4 of the children had high school or higher education. It was determined that 2 of the children in the control group were the only child in the family, 2 had two siblings, and 4 had three or more siblings.

Measures

Personal information form

Mothers of the children participating in the program were asked to fill the personal information form, which was generated to gather information on the ages and gender of the children, the number of children in the family, the ages of the parents, and their levels of education.

Test of early mathematics ability third edition (TEMA-3)

The Early Mathematics Ability Test - 3 (TEMA-3) (Ginsburg and Baroody, 2003) was used to assess children's mathematical skills. The scale measures the mathematics ability of children between the ages of 3 and 8 years and 11 months. It contains items designed to measure formal and informal mathematical knowledge. The math score of the children is determined by converting their raw score on the scorecard according to their chronological age. An increase in the math score indicates an increase in the children's math ability. The internal consistency scores of the tool were found to be 0.92 or higher (Ginsburg & Baroody, 2003). TEMA-3 was adapted into Turkish by Erdoğan (2006). The test-retest correlation calculated for reliability was calculated as 0.90. The internal consistency coefficient was found to be 0.92 for Form A (Erdoğan, 2006).

Child parent relationship scale

The Child-Parent Relationship Scale, which was developed by Pianta (1992) and adapted into Turkish by Akgün and Yeşilyaprak (2010), was used to determine the quality of the relationship between pre-school children and their parents before and after the implementation of the program. Regarding the reliability of the scale, the internal consistency coefficients (Cronbach's alpha) were found to be 0.85 for the conflict dimension subscale, 0.73 for the positive dimension subscale and 0.73 for the total scale.

Mother interview form for the number and operation training program

Mother Interview Form was designed to understand mothers' perspectives on the training program. Items on the form address mothers' expectations of the training program, the implementation process, meeting their expectations after the implementation, and the general contributions of the program to children’s mathematics ability. The form consists of 10 unstructured questions.

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Analysis Plan

Quantitative and qualitative data were collected for the research. Descriptive statistics (minimum, maximum, mean, standard deviation) were used in the analysis of the data. In addition, the Mann Whitney-U Test, which is one of the nonparametric tests used in experimental studies, especially when the sample size is low and the normality assumption is not met, and the Wilcoxon Signed Rank Test, which is used in cases where it is tried to determine whether there is a significant relationship between the pretest and posttest scores, is the parametric analysis of two-way analysis of variance. Friedman Test, which is calculated as an alternative to non-existent, was used. Content analysis was used in the analysis of qualitative data. The quantitative data collection tools used in the research are Early Mathematics Ability Test-3 and Child Parent Relationship Scale. The qualitative data collection tool used in the research is Mother Interview Form for the Number and Operation Training Program.

Procedure

Data Collection

Preparation of the number and operation training program

The training program is designed for 48-65 month-old children who do not attend any pre-school education institutions and for the mothers of these children who also do not attend any family education programs. The program aims to improve the number and operation skills of children and the mother-child relationship. The preparation stages of the training program are given as follows:

- In the creation of the content of the training program, a literature review on early childhood mathematics education was conducted. In this direction, firstly, the achievements and indicators established by the Ministry of Education - Preschool Education Program (2013) and the mathematics standards determined by NCTM (National Council of Teachers of Mathematics) (2000) for the pre-school period were examined.
- Secondly, mathematics education programs and some projects prepared for the school and home environment for the acquisition of mathematics skills in the pre-school period in Turkey and abroad were examined.
- Thirdly, the number and operation skills that are expected to be acquired in pre-school period and the basic skills associated with these skills were determined in accordance with the purpose of the educational program. These skills are defined as "matching, counting, recognizing numbers, classification, comparison, problem solving, addition and subtraction".
- Subsequently, the contents of the 13 home visits, which make up the training program were included in the program.
- A total of 5 experts evaluated the contents of the program and the appropriateness of the activities prepared for the program in terms of "acquisition and indicator, instructions, material properties, the subjects discussed and the evaluation".
- According to the feedback received, the program was given its final form.

Pre-Tests

Personal Information Form, Early Mathematics Ability Test-3 (TEMA-3), Child-Parent Relationship Scale were implemented as pre-tests. In addition, Mother Interview Form was applied to understand the perspectives of the mothers on the purpose of their participation in the Number and Operation Training Program.

Implementation of the Number and Operation Training Program

In the first week of the training program, all the mothers and their children were visited in their home environment, and the mothers were informed about the contents of the training program. After an introductory game, the days and hours of the visit were determined with each mother, taking into account the availabilities of the mothers. The families were informed that the researcher would visit their homes once a week for 13 weeks and the home would last an average of 45 minutes. Each week, the mothers were asked to prepare the necessary materials for the following week. Mothers were asked to repeat the activity with the participation of other family members (father, sibling, and so on) during the week.

Post-Tests and Permanence Tests

Early Mathematics Ability Test-3 (TEMA-3) and Child-Parent Relationship Scale were implemented as post-tests and permanence tests. In addition, Mother Interview Form was used.

Results

The data collected in this part of the study are analyzed and the results are presented in the following tables.

Findings on Children's Early Math Ability

Table 1
Descriptive Statistics Calculated on the Early Mathematics Abilities of the Children in the Experimental and Control Groups before the Implementation.

Tests	Groups	*N*	Min.	Max.	*M*	*SD*
Pre-test	Experimental	13	77.00	124.00	92.46	11.48
	Control	8	75.00	118.00	93.88	13.37

As demonstrated in Table 1, the lowest score in TEMA-3 received by the children in the experimental group before the implementation was 77 while the highest score was 124. The average score of the children in the experimental group on the Early Mathematics Ability Test was 92.46, with a standard deviation of 11.48. It was determined that the scores of the children in the control group on the Early Mathematics Ability Test ranged between 75 and 118. The average score of the children in the control group on the Early Mathematics Ability Test before the implementation was 93.88, with a standard deviation of 13.37. According to the categorization made in line with the points that can be obtained from TEMA-3, scores within 90-110 range indicate the average skill level (Gingsburg & Baroody, 2003).

Table 2

Mann Whitney U Test Results on the Early Mathematics Ability Pre-Test Scores of Children in Experimental and Control Groups.

Test	Group	N	Mean Rank	Sum of Ranks	U	z	p
Pre-test,	Experimental	13	10.62	138.00			
	Control	8	11.63.	93.00	47.000	0.364	0.716

$^*p < 0.05$

As shown in Table 2, the average scores of the children in the experimental and control groups before the implementation of the program did not show a significant difference (z = 0.364; p > 0.05).

Table 3

Mann Whitney U Test Results on Early Mathematics Ability Post-Test Scores of Children in Experimental and Control Groups.

Test	Group	N	Mean Rank	Sum of Ranks	U	z	p
Post-test	Experimental	13	13.77	179.00			
	Control	8	6.50	52.00	16.000	2.618	0.009*

$^*p < 0.05$

As indicated in Table 3, the average scores of the children in the experimental and control groups on the Early Mathematics Ability post-test show a significant difference ($z = 2.618$; $p < 0.05$).

Table 4

Friedman Test Results on Early Mathematics Ability Pre-Test, Post-Test and Permanence Test Scores of Children in Experimental Group.

Group	Test	N		X^2	p
Experimental Group	Pre-test	13	1.08		
	Post-test	13	2.65	21.256	0.000*
	Permanence Test	13	2.27		

$^*p < 0.05$

As demonstrated in Table 4, it is determined that the scores of the children in the experimental group on the Early Mathematics Ability Test before the implementation, after the implementation and in the permanence practice indicate a significant difference ($X^2 = 21.256$; $p < 0.05$).

Table 5 shows that the Early Mathematics Ability post-test scores and pre-test scores of the children in the experimental group differ significantly (z = 3.062; $p < 0.05$). When the mean and total scores are examined, it is seen that the post-test scores of the children in the experimental group are higher than the pre-test scores. Based on the results in Table 5, the Early Mathematics Ability permanence test and pre-test scores of the children in the experimental group also differ significantly (z = 3.065; $p < 0.05$). An examination of the mean rank and total scores shows that the permanence test scores of the children in the experimental group are higher than the pre-test scores. In line with the information in Table 5, it is found that there is no significant difference between the Early Mathematics Ability post-test average score and permanence test average score of the children in the experimental group (z = 1.205; $p > .05$).

Table 5

The Results of the Wilcoxon Signs Test Regarding the Early Mathematics Ability Pre-Test, Post-Test and Permanence Test Scores of the Children in the Experimental Group.

Group	Test		N		Rank Sum	z	p
Experimental Group	Post-test- Pre-test	Negative Ranks	0	0.00	0.00		
		Positive Ranks	12	6.50			
		Ties	1		78.00	3.062	0.002*
	Permanence Test-Pre test	Negative Ranks	0	0.00	0.00		
		Positive Ranks	12	6.50			
		Ties	1		78.00	3.065	0.002*
	Permanence Test- Post-test	Negative Ranks	6	3.50	21.00		
		Positive Ranks	1	7.00			
		Ties	6		7.00	1.205	0.228

$^*p < 0.05$

Table 6

The Results of the Wilcoxon Signs Test Regarding the Early Mathematics Ability Pre-Test and Post-Test Scores of the Children in the Control Group.

Group	Test	Rank	N	Mean Rank	Rank Sum	z	p
Control Group	Post-test- Pre-test	Negative Rank	4	3.50	14.00		
		Positive Rank	3	4.67	14.00	0.001	0.999
		Ties	1				

*$p < 0.05$

As demonstrated in Table 6, there is no significant difference between the pre-test and post-test results on the early mathematics abilities of the children in the control group ($z = 0.001$; $p > 0.05$).

Findings Regarding the Mother and Child Relationship

The Mother-Child Relationship Test has two sub-dimensions: conflict relationship and positive relationship. The lowest score that can be obtained from the 14 items in the conflict relationship sub-dimension of the scale is 14 while the highest score is 70. High scores indicate a high conflict relationship between children and their mothers. Of the 10 items in the latter sub-dimension, designed to determine the positive relationship between pre-school children and their mothers, the lowest score that can be obtained is 10 while the highest score is 50. High scores from the positive relationship sub-dimension indicate a high positive relationship between children and their mothers.

Table 7

Descriptive Statistics Calculated Regarding the Mother-Child Relationship Test Scores of the Children in the Experimental and Control Groups before the Implementation.

Sub Dimension	Group	N	Min.	Max.	*M*	*SD*
Conflict Relationship	Experimental	13	29.00	47.00	36.77	6.07
	Control	8	29.00	50.00	38.13	6.92
Positive Relationship	Experimental	13	34.00	48.00	42.00	4.26
	Control	8	30.00	47.00	39.63	4.84

As revealed in Table 7, before the implementation of the program, the conflict relationship scores of the children in the experimental group varied between 29 and 47, with an average score of 36.77 (±6.07); while the scores of the children in the control group in this sub-dimension ranged between 29 and 50, with an average of 38.13 (±6.92). Thus, it can be concluded that the conflict relationship between the children with their mothers in both the experimental and control groups were moderate before the implementation of the program. Table 7 shows that the pre-test scores of the children in the experimental group on the positive relationship sub-dimension ranged between 34 and 48, with an average of 42.00 (±4.26). The positive relationship pre-test scores of the children in the control group with their mothers varied between 30 and 47, with an average score of 39.63 (± 4.84). In this context, the average scores calculated before the implementation indicate that the children in both the experimental and control groups had a positive relationship with their mothers, in general.

In accordance with the information in Table 8, it is determined that the mother conflict relationship pre-test scores of the children in the experimental and control groups do not show a significant difference ($z = 0.472$; $p > 0.05$). Similarly, it is found that the positive relationship between the children with their mothers in the experimental and control groups did not show a significant difference before the implementation of the program ($z = 1.163$; $p > 0.05$).

Based on the information in Table 9, the conflict relationship scores of the children in the experimental and control groups with their mothers do not show a significant difference after the implementation ($z = 0.653$; $p > 0.05$). As shown in the table, however, the positive relationships of children with their mothers differ significantly between groups ($z = 2.760$; $p < 0.05$). After the implementation, it is found that the children in the experimental group have a higher level of positive relationship with their mothers than the children in the control group.

As demonstrated in Table 10, the conflict relationship scores of the children in the experimental group before and after the implementation do not show a significant difference ($z = 0.254$; $p > 0.05$). In other words, it is determined that the mother conflict relationship pre-test and post-test scores of the children in the experimental group are similar. However, it is found that the positive relationship between the mothers and the children in the experimental group and the pre-test and post-test scores show a significant change ($z = 0.254$; $p < 0.05$). An examination of the mean rank and total scores indicates that the post-test scores of the children in the experimental group on the positive relationship sub-dimension are higher than the pre-test scores.

Table 11 displays that the conflict relationship post-test and permanence test scores of the children in the experimental group are significant different ($z = 1.994$; $p < 0.05$). Based on the mean rank and total scores, it is found that the scores on the conflict relationship of the children with their mothers are lower in the post-test than in the permanence test. It ıs found that the positive relationship permanence test and post-test scores of the children in the experimental group also show a significant change ($z = 2.871$; $p < 0.05$).

According to the information contained in Table 12, it is found that conflict relations with the mothers of children in the control group do not differ significantly

Table 8

Mann Whitney U Test Results Calculated for the Mother-Child Relationship Pre-Test Scores of the Children in the Experimental and Control Groups.

Sub Dimension	Group	N	Mean Rank	Rank Sum	U	z	p
Conflict Relationship	Experimental	13	10.50	136.50	45.500	0.472	0.645
	Control	8	11.81	94.50			
Positive Relationship	Experimental	13	12.23	159.00	36.000	1.163	0.268
	Control	8	9.00	72.00			

*p < 0 .05

Table 9

Mann Whitney U Test Results Calculated for the Mother-Child Relationship Post-Test Scores of the Children in the Experimental and Control Groups.

Sub Dimension	Group	N	Mean Rank	Rank Sum	U	z	p
Conflict Relationship	Experimental	13	11.69	152.00	43.000	0.653	0.547
	Control	8	9.88	79.00			
Positive Relationship	Experimental	13	13.92	181.00	14.000	2.760	0.005*
	Control	8	6.25	50.00			

*p < 0 .05

Table 10

The Results of the Wilcoxon Sign Test Calculated Regarding the Mother-Child Relationship Pre-Test and Post-Test Scores of the Children in the Experimental Group.

Sub Dimension	Test	Rank	N	Mean Rank	Rank Sum	z	p
Conflict Relationship	Post-test-Pre-test	Negative Rank	3	5.17	15.50	0.254	0.799
		Positive Rank	4	3.13	12.50		
		Ties	6				
Positive Relationship	Post-test-Pre-test	Negative Rank	1	1.50	1.50	2.120	0.034*
		Positive Rank	6 6	4.42	26.50		
		Ties					

*p < 0 .05

Table 11

Wilcoxon Signs Test Results Regarding the Mother-Child Relationship Post-Test and Permanence Test Scores of the Children in the Experimental Group.

Sub Dimension	Test	Rank	N	Mean Rank	Rank Sum	z	p
Conflict Relationship	Permanence Test- Post-test	Negative Rank	11	5.67	17.00	1.994	0.046
		Positive Rank	2	7.40	74.00		
		Ties	0				
Positive Relationship	Permanence Test- Post-test	Negative Rank	3	7.86	86.50	2.871	0.004
		Positive Rank	10	2.25	4.50		
		Ties	0				

*p < 0 .05

Table 12

Results of the Calculated Wilcoxon Signs test Regarding the Mother-Child Relationship Pre-Test and Post-Test Scores of the Children in the Control Group.

Sub Dimension	Test	Rank	N	Mean Rank	Rank Sum	z	p
Conflict Relationship	Post Test- Pre-test	Negative Rank	3	2.00	6.00	1.604	0.109
		Positive Rank	0	0.00	0.00		
		Ties	5				
Positive Relationship	Post Test- Pre-test	Negative Rank	3	2.67	8.00	1.105	0.269
		Positive Rank	1	2.00	2.00		
		Ties	4				

*$p < 0.05$

before and after the implementation (z = 1.604; p > 0.05). In other words, the mother conflict relationship pre-test and post-test results of the children in the control group are similar. The positive relationships of the children in the control group with their mothers do not show a significant change before and after the implementation (z = 1.105; p > 0.05).

Findings Regarding Mothers' Views on Number and Operation Training Program

The findings obtained from the answers given by the mothers to the interview form are given below.

Mothers' Views Regarding the Purpose of Participation in the Number and Operation Training Program.

In the interviews, the mothers stated that they aimed to participate in the training program both for themselves and their children. Twelve mothers expected the program to "contribute positively to their relationship with their children", eight mothers aimed to "learn new games", five mothers indicated that the program would allow them to "spend quality time with their children" and two mothers expected it to be a "fun activity and a way of having a pleasant time". Additionally, 13 mothers stated that their children would "learn numbers", eight mothers expressed that their children would "learn new games and learn addition and subtraction", five mothers participated in the education program for their children to play games with other family members, and two mothers participated in the program to prepare their children for the primary school. The following quotes from the mothers indicate their main purposes of participation in the program.

> *"I wonder if my child will be confused because he is young. But I've never played an educational game with my kid before. Maybe I can learn educational games." (M-2)*

> *"Having a pleasant, fun time with my child while playing the games you (educator) teach will contribute to our relationship." (M-11)*

> *"I want my child to develop math skills and go to kindergarten prepared. I want to spend quality time with my child so that he can have fun and not get bored." (M-13)*

> *"I, at least, want my child to play more with her sister and learn something, instead of watching TV at home." (M-5)*

Mother's Views Regarding the Achievements of their Children After the Implementation of the Number and Operation Training Program.

Thirteen mothers stated that their children "learned numbers and new games" and "played the game they learned with other family members". In addition, six mothers mentioned that their children "learned geometric shapes, understood the logic of cardinal numbers and recognized numbers". Four mothers stated that their children "had an idea about addition" and three mothers said that their children "had an idea about subtraction". Two mothers also expressed that their children could "match objects and numbers and their children learned to divide", and one mother indicated that her child "had an idea about the relative operation of numbers." In addition, two mothers stated that their children could "match objects and numbers", two mothers pointed out that their children could "apportion", and one mother expressed that her child "understood how close the numbers are to each other". Below are some of the responses of the mothers:

> *"Of course, my child has improved a little. He never knew the numbers before. Actually, he wasn't interested. Now he can count from 1 to 10. Although he mixes them up, he knows the numbers. He also learned to do addition within the numbers up to 10." (M-9)*

> *"My child says 'Look, mom, there is a spoon. One, two'. S/he says 'there was one, now there are two. One, two. There are two'. Our communication has increased thanks to the materials you (educator) brought in the training program and the activities we carried out." (M-8)*

> *"We are now counting many things that come across. He knows the door numbers we see. He counts*

spoons and forks in the kitchen. He helps me set the table. He asks "How many of us are there? How many forks should I bring?" (M-3)

"She learned how to count numbers rhythmically both forwards and backwards, match numbers with objects, add and subtract with objects." (M-1)

"My child learned to add and subtract. He learned which number is bigger and which number is smaller. He learned the proximity and distance of the numbers to each other." (M-10)

Mothers' Views Regarding Achievements from their Perspective after the Implementation of the Number and Operation Training Program.

Thirteen mothers stated that they learned educational games to play with their children after the implementation of the training program, seven mothers expressed that they spent more quality time with their children, and six mothers stated that the positive relationship with their child increased. Below are some of the mothers' statements:

"We had never played an educational game before. We had so much fun playing with the hopscotch carpet. We matched the numbers with the lids. My son got more enthusiastic as he got to know the numbers." (M-9)

"The games were so much fun. My daughter would be bored if it didn't involve playing together. Because she's a kid who gets bored easily. But we played the games together after you. We had fun." (M-13)

"It definitely contributed positively to my relationship with my child. We played more games. She also played with his brother and his father." (M-10)

General Opinions of the Mothers on the Number and Operation Training Program.

Thirteen mothers stated that the education program "contributed to the children's mathematical skills", 12 mothers said it "positively affected their relationship with their children", 10 mothers expressed that they "had an idea about how to support their children's mathematical skills" and 10 mothers stated that "it was good for the educator to come to the house". In addition, nine mothers mentioned that "they had the opportunity to see what their children could do in mathematics" and that "they played games not only with each other but also with other family members". The following quotations demonstrate the mothers' perspectives:

"My child and I now count the cars or bikes outside. Sometimes, he mixes the numbers up, but I help him. He's counting much better than before." (M-4)

"My child is now very eager to count. As we place the dishes, we count the forks and spoons. And now I want him to count things every chance he gets." (M-1)

Discussion

This study shows that the Number and Operations Training Program that necessitated the active participation of mothers in the home environment is effective on children's early mathematics skills and that the effects of the program are permanent. The permanence of the training program was determined by the permanence tests performed 4 weeks after the application of the post-tests. In a similar study conducted by Niklas et al. (2016), it was found that children in the experimental group showed more improvement in numerical competence than children in the control group. Several similar studies demonstrated a relationship between the learning environment at home and children's numerical skills (Ciping et al., 2015; Kleemans et al., 2012; Kluczniok, 2017; Skwarchuk et al., 2014). Hwang (2020) found in his study that students who engage in math activities at home earlier are more likely to have higher math achievement in fourth grade. In addition, considering that parents' beliefs about academic abilities and their children's academic abilities (Zippert & Rittle-Johnson, 2018) have an impact on children's educational investments (Dizon-Ross, 2019), programs such as the Number and Operations Training Program will contribute to the development of children by increasing parents' awareness of mathematics.

This study demonstrates that in addition to the early mathematics ability scores of the children in the experimental group, the positive relationship scores between the mothers and children also improved. In the training program, children had the opportunity to make mathematical conversations as well as spend more quality time with their mothers by engaging in different activities such as playing number games, playing with puppets, playing hopscotch and making cookies. Previous research also showed that children's problem solving and social skills are promoted through programs that incorporate activities suited to children's developmental characteristics, and through mothers' active participation by providing a stimulating environment for their children (Kılınç & Aral, 2015). According to Watts and Broaddus (2002), in a game-based mother-child education, mothers learn to improve their relationship with their children, thus contributing to children's personal development.

It can be said that the puppets, hopscotches, number cards, geometric shapes, cookie recipes brought to the home environment and informative conversations with mothers about the development of early mathematics skills within the scope of the Number and Operation Training Program contribute to the organization of the home environment in a way that supports the development of children. In a similar study by Vandermaas-Peeler et al., (2012), which examined the arithmetic interactions of four-year-old children with their parents during home cooking activities, it was determined that parents of children in the experimental group created more opportunities for their children to use advanced mathematics. From this point of view, it can be said that home-centered interventions can be effective in improving the mathematics skills of children by informing parents about mathematics implementation (Sonnenschein et al., 2016). In another similar study, Şahin (2008) concluded that the concept acquisition scores of children in the experimental group to which the Toy Focused Home Education Program was applied were significantly higher than the control group. When these findings are evaluated together, participation in mathematics-related activities at home (Berkowitz et al., 2015; Clements & Conference Working Group, 2004; Sonnenschein et al., 2014; Peeters et al., 2012) and the frequency of math talk with parents (Hojnoski et al., 2014; Lukie et al., 2014; Ramani et al., 2015; Skwarchuk, 2009; Starkey & Klein, 2000; Uscianowski et al., 2020; Zippert & Ramani, 2017) can be said to contribute to pre-school children's numerical knowledge and skills.

In the investigation of mothers' views, the main achievements of the training program are found to be children's recognizing and learning numbers, sharing, adding and subtracting, learning geometric shapes, learning the total number of objects shown in a group, the distance between numbers, playing new games and playing with family members. Thus, it is vital for children to engage in mathematical activities in the home environment with their parents and for parents to create opportunities to use mathematical language in order to support children's development of early mathematical skills.

According to mothers who participated in the training program, the program seems to be beneficial for them in terms of "spending more time with their children, learning new games and playing these games with other family members". Through activities, mothers learned educational games that contributed to their quality time with their children. In a similar study, Mayer et al. (2015) found that parents who participated in the program used the readings provided by the educators to spent more time with the children after the implementation of the program. When these results are evaluated together, it can be concluded that activities that parents and children do together and the materials provided to them have a positive effect on both the parent-child relationship and children's mathematical skills.

In conclusion, this research finds that the Number and Operations Training Program, which was implemented with the active participation of mothers in the home environment, increased children's early mathematics skills, improved the mother-child relationship, and raised mothers' awareness of strategies that would directly support children's mathematics skills in the home environment.

Limitations and Future Directions

There are a few limitations to the present study that should be indicated. First, the scope of the measures (i.e. TEMA-3, Child Parent Relationship Scale (CPRS), Mothers' Interview Form) is limited by the items and responses. Second, the sample of the study consists of children between the ages of 48-65 months, who show normal development, and their mothers in Aksaray Province (in Turkey).

In line with these limitations, the following suggestions can be made:

- In the study, a home-based mathematics training program was designed for pre-school children. Longitudinal studies can be conducted to observe the mathematics achievements and mother-child relationships of children during their primary school years.
- Home-based early mathematics intervention programs can be designed that necessitate the active participation of fathers in order to improve children's early mathematics skills and father-child relationships.
- The training program can be applied to a wider sample of children from different age groups and can involve families from different socio-economic statuses and educational levels.
- Different programs can be applied in the home environment to support the development of children of different age groups.

Conflict of Interest: Authors have no conflict of interest to declare. The data collection process was completely voluntary and no monetary incentives was offered for the participation of the respondents.

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Appendix

**Methods and Techniques Used:* During the implementation of the Home-Centered Number and Operations Training Program, meeting, demonstration and role playing techniques were used.

**The home visit to be made within the scope of the "Home-Centered Number and Operations Training Program" and the targeted math skills are given below:*

1. Home Visit: Meeting Activity

2. Home visit: One-to-one correspondence, counting (from 1 to 5), comparison

3. Home visit: One-to-one correspondence, counting (from 1 to 10), number recognition

4. Home visit: Counting (from 1 to 10), number recognition, classification, problem solving, comparison

5. Home visit: Counting (from 1 to 10), recognizing numbers, One-to-one correspondence, problem solving

6. Home visit: Counting (from 1 to 10), operation (addition), problem solving

7. Home visit: Counting (from 1 to 10), operation (addition), problem solving

8. Home visit: One-to-one correspondence, counting (from 1 to 10), number recognition

9. Home visit: Counting (from 1 to 10), operation (subtraction), problem solving

10. Home visit: Recognizing numbers, problem solving

11. Home visit: Counting (from 1 to 10), classification, comparison

12. Home visit: One-to-one correspondence, counting (from 1 to 10), operation (addition, subtraction)

13. Home visit: Operation (addition), problem solving, number recognition, classification

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Predicting Mathematical Learning Difficulties Status: The Role of Domain-Specific and Domain-General Skills

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Abstract

This study investigated which domain-specific and domain-general skills measured at grade 1 predict mathematical learning difficulties (MLD) status at grade 3. We used different cut-off criteria and measures of mathematics performance for defining the MLD status. Norwegian children's (N = 206) numeracy, cognitive, and language skills were measured at grade 1 and arithmetic fluency and curriculum-based mathematics (CBM) at grade 3. Logistic regression analyses showed that symbolic numerical magnitude processing, verbal counting, and rapid automatized naming predicted MLD25 status (performance ≤ 25th percentile) based on arithmetic fluency, whereas verbal counting skills and nonverbal reasoning predicted the status based on CBM.
The same predictors were found for MLD10 status (performance ≤ 10th percentile), and in addition, rapid automatized naming also predicted the status based on CBM. Only symbolic numerical magnitude processing and verbal counting predicted LOW status (performance between 11–25th percentile) based on arithmetic fluency, whereas nonverbal reasoning and working memory predicted LOW status based on CBM. Different cut-off scores and mathematics measures used for the definition of MLD status are important to acknowledge, as these seem to lead to relatively significant variation in which students are identified as having MLD and which factors contribute to the MLD status.

Keywords:

Arithmetic, Counting, Mathematical Learning Difficulties, Nonverbal Reasoning, Rapid Automatized Naming

Introduction

Individual differences in mathematics learning and performance in the beginning of schooling are well-acknowledged (Aunio & Niemivirta, 2010; Jordan et al., 2009; ten Braak et al., 2022; Zhang et al., 2020). Children in early grades are rarely formally diagnosed as having developmental learning disorder in mathematics, also called dyscalculia, until the effect of teaching has been taken into account (ICD-11; World Health Organization,

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2019). Nonetheless, we can reliably identify students who perform weaker in mathematics than their peers and show difficulties in their mathematics learning. In research literature, different cut-off criteria for performance, ranging from 10th to 35th percentile, have been used to identify students at risk for or having mathematical learning difficulties (MLD) (e.g., Aunio et al., 2021; Kroesbergen & van Dijk, 2015; Lin et al., 2021; Mazzocco et al., 2013). A wide cut-off range used for the definition of MLD leads to an inclusion of a variety of performance within MLD; the lower end showing more severe difficulties than the upper end. Not until recently have the researchers started to show more consensus on the terms and cut-off criteria of MLD. Individuals performing at the lowest 10th percentile are commonly referred to as having mathematical disability/disorder or developmental dyscalculia, whereas those performing between the 11th and 25th percentile are referred to as low-performing/achieving (Geary, 2011). The term mathematical learning difficulties, independent of the severity of MLD, can then be used as an umbrella term for all those performing at or below the 25th percentile.

In addition to the severity criteria, different types of mathematics measures are used in research to identify students with MLD. Measures of arithmetic performance are typically applied (Cañizares et al., 2012; Koponen, Aro, et al., 2018), as poor arithmetic fluency is one of the main characteristics of students having MLD (Geary, 2011; Gersten et al., 2005). Alternatively, broad mathematics performance measures are used (Jordan et al., 2002). Sometimes the definition of MLD includes a persistence criterion, meaning that the student needs to perform low in at least two consecutive time points (Mazzocco et al., 2013; Stock et al., 2010). Currently, we have little knowledge of the extent to which the different mathematics measures overlap in defining MLD. That is, would a brief arithmetic fluency measure identify the same students as a broader mathematics performance test?

Researchers have also been curious about which domain-specific and domain-general skills are related to or may contribute to MLD, and whether these differ between the subgroups of MLD (Geary et al., 2012; Huijsmans et al., 2022; Salihu & Räsänen, 2018). A set of domain-specific skills, such as numerical magnitude processing (Cañizares et al., 2012; De Smedt & Gilmore, 2011) and counting skills (Koponen et al., 2019), and domain-general skills, such as working memory (Menon, 2016; Passolunghi & Mammarella, 2010) and rapid automatized naming (Van Luit & Toll, 2018), have been found to be related to MLD. However, drawing conclusions about their predictive role becomes complicated due to the different cut-off criteria for MLD and various measures used for the identification of students with MLD.

On one hand, it has been suggested that deficit in either numerical magnitude processing (NMP) (Butterworth, 2005) or accessing the magnitude in symbols (Rousselle & Noël, 2007) underlies the most severe MLD (but see Mammarella et al., 2021 for no evidence for a core deficit in MLD). On the other hand, a persistent low performance in mathematics has been suggested to stem from having weaknesses in domain-general cognitive skills (e.g., working memory) or lacking mathematics motivation (Geary, 2011; Price & Ansari, 2013). By contrast, the double deficit model suggests that weaknesses in both NMP and working memory would be associated with the most severe MLD (Kroesbergen & van Dijk, 2015). While there are competing theories about the associations between domain-specific and domain-general skills and MLD, and whether these differ between the subtypes of MLD (Geary et al., 2012; Huijsmans et al., 2022; Tolar et al., 2016), the empirical evidence has mostly relied on cross-sectional data, thus focusing only on the concurrent relations between the predictors and MLD status (e.g., Cañizares et al., 2012; Passolunghi & Mammarella, 2010; Tolar et al., 2016; Van Luit & Toll, 2018; Willburger et al., 2008). Longitudinal data would seem more accurate in identifying the most relevant domain-specific and domain-general predictors of MLD status.

Our study adds to the current research by investigating which domain-specific and domain-general skills measured in the first grade predict MLD status in the third grade. The novelty of our study lies in taking into consideration different cut-off criteria and mathematics measures (i.e., arithmetic fluency and curriculum-based mathematics [CBM]) in defining the MLD. Further, we include simultaneously several domain-specific (i.e., symbolic numerical magnitude processing [SNMP] and counting skills) and domain-general skills (i.e., nonverbal reasoning, working memory, rapid automatized naming, and vocabulary) as predictors of MLD status. To examine if domain-specific and domain-general skills predict MLD status differently when using different cut-off criteria based on the sample-based percentiles, we first divide the students into two groups: those who perform at or below the 25th percentile are referred to as having mathematical learning difficulties (MLD25), while rest of the students are referred to as typically performing (performance over the 25th percentile). Next, we divide the MLD25 group into those who are showing more severe difficulties, namely mathematical learning disorder (performance at or below the 10th percentile, MLD10) and those who are low performing (performance between the 11th to 25th percentile, LOW). The status of MLD is based separately on arithmetic fluency and on CBM. This enables us to examine the overlap of MLD statuses based on arithmetic fluency and CBM (e.g., how many students are classified as MLD25 in both arithmetic fluency and

CBM), and whether predictors of MLD status vary as a function of mathematics measure (arithmetic fluency vs. CBM) and cut-off criteria (10th vs. 25th percentile).

Domain-Specific Skills as Predictors of MLD

Domain-specific skills in mathematics context mean different mathematical skills. Here, we focus on two such skills, numerical magnitude processing (NMP) and counting skills, which have shown to be related to or to predict later mathematics performance, concerning both the typically performing students and students with MLD. NMP is considered to be an innate ability, which enables individuals to approximately process numerical magnitudes (Dehaene et al., 1998). A traditionally used task for measuring NMP is to compare two sets of dots, and to quickly decide which side has more dots. Even if some studies have shown that students with MLD often have poorer NMP skills compared to their peers without MLD (Mazzocco et al., 2011; Mussolin et al., 2010), there is recent evidence showing that symbolic NMP (SNMP) would be even a better predictor of mathematics performance and more strongly associated with MLD than non-symbolic NMP (Cañizares et al., 2012; De Smedt & Gilmore, 2011; Desoete et al., 2012; Nosworthy et al., 2013). For this reason, we focus in this study on the role of SNMP. The tasks often used to measure SNMP are similar to NMP comparison tasks, but 1- or 2-digit numbers are used as stimuli instead of dots (Brankaer et al., 2017). While comparing the numbers, and to choose the bigger number, the student needs to access the magnitudes of those numbers. A slow response indicates a deficit in accessing and processing of the magnitudes (Rousselle & Noël, 2007). Concerning school beginners, Desoete et al. (2012) found that SNMP measured in kindergarten (5–6 years old) was a good predictor of procedural calculations (i.e., 34 + 21, or given in as "6 more than 48 is…") in the second grade. Further, they found that although children with MLD showed weakness in both non-symbolic and symbolic NMP in kindergarten, it was only in SNMP they showed a deficit in grade 2. Overall, the findings are still mixed when it comes to the role of (S)NMP and severity level of MLD. Some research supports that students with severe MLD are characterized with poor NMP (Mazzocco et al., 2011), while some research has not found a difference between the subgroups of MLD (i.e., MLD10 and LOW) (Huijsmans et al., 2022)

Counting skills (i.e., verbal number sequence skills and object counting) develop typically parallel with early arithmetic skills (i.e.., addition and subtraction), and can also be seen as prerequisites for learning arithmetic skills, because children often use these as their strategies in solving arithmetic calculation problems (Koponen et al., 2019). Typically, children with MLD use more immature, counting-based strategies, while their peers use retrieval strategies, that is, they quickly retrieve the answer from long-term memory (Ostad, 1998). Counting skills have been found to be associated with (Lopez-Pedersen et al., 2021) and to predict later arithmetic and mathematics performance (Aunio & Niemivirta, 2010; Koponen et al., 2019). Furthermore, children performing low (Hassinger-Das et al., 2014; Toll & Van Luit, 2014) and with severe MLD (Landerl et al., 2004) have shown weaker counting skills compared to their peers.

Domain-General Skills as Predictors of MLD

Several domain-general skills, such as working memory, executive functions, rapid automatized naming, and language, have been found to be associated with mathematics performance and development (Chu et al., 2016; Friso-van den Bos et al., 2013; Koponen, Eklund, et al., 2018; Purpura & Ganley, 2014), and with MLD (Mammarella et al., 2021; Passolunghi & Mammarella, 2010; Purpura et al., 2017; Van Luit & Toll, 2018). Further, separate domain-general cognitive profiles depending on the severity level of MLD has been suggested (Geary et al., 2012). However, in a recent study, Hujsman et al. (2022) did not find support for this. In their study, the severity of MLD did not result in differences in the cognitive profiles, and further, the cognitive profiles for mathematics development from fourth to fifth grade were rather similar between the students with MLD and typically performing students.

Nonverbal reasoning (also called as nonverbal or fluid intelligence) has been shown to be a consistent predictor of mathematics performance in different age groups of students (Kyttälä & Lehto, 2008; Pina et al., 2014). Good nonverbal reasoning skills are an advantage in solving mathematical problems, because students need to be able to make logical decisions and to proceed systematically with the task (e.g., entertain their solutions and if proven false try a new solution) (Engle, 2018). Students with MLD typically perform weaker in nonverbal reasoning, albeit within normal range, compared to their peers without MLD (Huijsmans et al., 2022).

Solving mathematics tasks requires working memory capacity for storing and manipulating information temporarily. Working memory and its different components (i.e., visuo-spatial sketchpad, phonological loop, central executive) have all been linked with mathematics performance (Friso-van den Bos et al., 2013). Students with MLD have been reported to show weaker working memory performance than their peers (Kroesbergen & van Dijk, 2015; Passolunghi & Mammarella, 2010). Huijsmans et al. (2022) found that those with low performance were characterized by difficulties in visual working memory, but interestingly, not those with severe MLD. However, working memory did not explain any variance in the mathematics development from fourth to fifth grade (Huijsmans et al., 2022). Our study, instead, focuses on the central executive component of working memory (i.e., manipulation of information). Prior research has shown that students with MLD have a deficit in their central executive functioning, and especially when central executive has been measured with numerical stimuli (Andersson & Lyxell, 2007; for a meta-analysis see David, 2012).

Rapid automatized naming (i.e., quickly naming familiar non-alphanumeric objects such as colors and figures, or alphanumeric objects such as letters or numbers) has been found to be more strongly related to arithmetic fact retrieval than to more general mathematics performance (for a meta-analysis see Koponen et al., 2017). Rapid automatized naming (e.g., naming a color) and fact retrieval (e.g., 5 + 3) both require quick access to and retrieval of phonological representations from long-term memory ("blue" and "eight", respectively) (Koponen et al., 2017). Concerning students with MLD, Donker et al. (2016) found that primary school-aged students had weaker performance in non-alphanumeric rapid automatized naming (e.g., colors) compared to their peers, but not in alphanumeric format (e.g., letters). Further, Mazzocco and Grimm (2013) found in their longitudinal study from kindergarten to grade 8 that those with LOW showed slight delays (i.e., slower response times) in naming of colors compared to their TYP peers, whereas those with severe MLD showed more persistent weakness in rapid naming of letters and colors. Based on prior findings that rapid naming of colors can be a good early predictor of mathematics performance and especially arithmetic fact retrieval (Koponen et al., 2017), and associated with MLD (Donker et al., 2016; Mazzocco & Grimm, 2013), in our study, we chose to have rapid naming of colors as a proxy for rapid automatized naming.

In early childhood, language skills have been linked with mathematics performance (Aunio et al., 2019), and especially mathematics related language to be a good predictor of low performance (Purpura et al., 2017), as well as influencing the development of early mathematical skills of LOW (Toll & Van Luit, 2014). In our study, we focus on vocabulary (expressive) in general, which has shown to play a role in children's mathematics learning (LeFevre et al., 2010). However, also conflicting results have been reported among school beginners, that is, no connection between vocabulary (receptive) and mathematics performance (Chow & Ekholm, 2019). Vocabulary is needed not only to understand mathematics teaching in general, but also to communicate using different mathematics concepts (e.g., comparison words, number words, geometrical object, words for operations [e.g., plus, minus]. Support for the importance of language in mathematics learning comes from studies, which have included students with developmental language disorder. In general, these students have consistently shown weaker mathematics performance compared to their peers in mathematical tasks that require expressing or understanding of language (e.g., verbal number sequences, counting of objects, arithmetic) while they have shown similar performance to their TYP peers in mathematics tasks with less demand on language (e.g., NMP, conceptual mathematics tasks) (for a review see Cross et al., 2019). Not many studies have investigated the role of different components of language among students with MLD. However, recently, Chow et al. (2021) showed that students with MLD (performance below 20th percentile on arithmetic fluency) performed lower than their peers in receptive vocabulary, morphology, and syntax.

Present Study

The present study expands on previous research by investigating how domain-specific and domain-general skills measured in the first grade predict MLD status among third graders when different cut-off criteria and measures of mathematics performance are used. Our research questions are as follows:

(RQ1) What is the overlap of MLD statuses based on arithmetic fluency and curriculum-based mathematics?

(RQ2) How do domain-specific (i.e., symbolic numerical magnitude processing and verbal counting skills) and domain-general (non-verbal reasoning, rapid automatized naming, working memory, and vocabulary) skills predict MLD status when using the 25th percentile cut-off criterion (MLD25) based on either arithmetic fluency (RQ2.1) or curriculum-based mathematics (RQ2.2)?

(RQ3) How do domain-specific and domain-general skills predict MLD status when further dividing the MLD25 into MLD10 (≤ 10th percentile) and low performers (LOW; 11–25th percentile) based on either arithmetic fluency (RQ3.1) or curriculum-based mathematics (RQ3.2)?

Although we have limited evidence available to strongly guide our hypothesis for RQ1, we expect relatively high overlap of MLD statuses based on different mathematics measures, but still distinct to a certain degree, as the mathematics content in arithmetic fluency is much more limited than in the broad CBM measure (H1).

Based on prior research, we hypothesize that all domain-specific and domain-general skills under investigation are likely predictors of MLD25 (H2), as this group encompasses those with more severe learning difficulties (MLD10) and milder learning difficulties (LOW). Regarding MLD25 based on arithmetic fluency, we expect SNMP (Desoete et al., 2012), counting skills

(Hassinger-Das et al., 2014; Landerl et al., 2004) and rapid automatized naming (Koponen et al., 2017; Mazzocco & Grimm, 2013) to be significant predictors (H2.1). Since the tasks in CBM have more variety and complexity regarding their mathematics content and procedures, thus requiring logical reasoning (Engle, 2018), executive functioning (David, 2012), and understanding task related vocabulary (Chow et al., 2021), we anticipate MLD25 based on CBM to be predicted by counting skills (Hassinger-Das et al., 2014), nonverbal reasoning, working memory, and vocabulary (H2.2).

Similar to MLD25 based on arithmetic fluency, we expect the two domain-specific skills and rapid automatized naming to predict both MLD10 and LOW status (H3.1). As to the status based on CBM, we presume domain-general skills, especially concerning working memory (David, 2012; Huijsmans et al., 2022), to exhibit different predictions on MLD10 and LOW (David, 2012; Huijsmans et al., 2022). Further, we expect the significant predictors to include counting skills (Hassinger-Das et al., 2014; Landerl et al., 2004), nonverbal reasoning (Huijsmans et al., 2022), and vocabulary (Chow et al., 2021) (H3.2).

Method

Participants

The current study is part of a research project that follows Norwegian children's numeracy development from first to third grade. Here, we use data from its first (grade 1, t1) and last (grade 3, t2) measurement time points. The final sample of participants was 206 children (M_{age} = 6 y. 9 m., *SD* = 3.4 m., girls 49%), from four schools located in the Oslo region, and who had data available from both time points. Due to Covid-19 restrictions in schools in spring 2021, 27 children from the initial sample of 265 were not able to participate in t2, and 32 children had either moved away or were absent from school on the data collection day. An ethical approval was given by the Norwegian Centre for Research Data before the data collection started, and consents for the participation were given by children's legal guardians.

Measures

Third-grade mathematics performance

Arithmetic fluency was measured using a standardized arithmetic test Regnefaktaprøven (Klausen & Reikerås, 2016). Children have 2 minutes to solve as many addition problems as possible out of 45 on one sheet, and same for subtraction. As a proxy of arithmetic fluency, we combined the sum scores of each subtest, thus the maximum possible score being 90 points.

A curriculum-based mathematics (CBM) test was developed in the project (Mononen, 2021) to measure children's overall performance in mathematics taught in grade 3. This paper-pencil group-based test includes 49 items from the topics of numbers (number sequences, comparison of multi-digit numbers), measurement (volume, length, money), calculations (multiplication facts, addition and subtraction algorithms) and fractions, and follows the learning outcomes set for the third grade in the national mathematics curriculum (ref.). Each task was instructed for the children and children worked with the tasks independently for 20–25 minutes. Each correctly solved item gave one point.

First-grade domain-specific numeracy skills

Symbolic numerical magnitude processing was measured using the 1-digit subtest of the SYMP test (Brankaer et al., 2017). In this paper-pencil test the child has 30 seconds to compare as many 1-digit number pairs as possible out of 60, by choosing the bigger number. One point is given for a correct answer, thus the maximum score being 60.

Verbal counting skills were measured using a normed Finnish LukiMat subscale (Salminen & Koponen, 2011), which was translated into Norwegian. The child was asked to orally count number sequences forwards and backwards, in steps of 1, 2, 5, and 10. Each correctly given number sequence gave one point, the possible maximum total score being 29 points.

First-grade domain-general skills

Nonverbal reasoning skills were measured using Raven's progressive colored matrices (Raven et al., 1990). The child chooses one of the six alternative pieces that fits the picture. One point was given for each correct answer, the maximum possible score being 34, as two first items were practice items.

Working memory was measured using a digit span backwards subtest from WISC-V (Wechsler, 2017). Digit span backwards captures the central executive component of WM, as modelled by Baddeley (Baddeley & Logie, 1999). Each digit span, ranging from 2–8 numbers, had two tasks, except for 2- and 3-digit spans having 4 tasks each. Each digit span was presented orally to the child forwards and the child needed to repeat the digits backwards. Following the test guidelines, the test was stopped if the child could not give a correct answer for both tasks with the same number of digits. The maximum total score was 18 points.

Rapid automatised naming was measured using a colors subtest from the Clinical Evaluation of Language Fundamentals (CELF-4) (Semel et al., 2003). The child needs to name 36 colored dots (including colors "gul" [yellow], "grønn" [green], "blå" [blue], "rød" [red]) as accurately and fast as possible. For the purpose of statistical analyses, we created a composite score, in which the correct number of named colors was divided by used time, and multiplied by ten.

Vocabulary was measured using a subtest 'Ordforståelse' of WISC-V (Wechsler, 2017) targeting expressive vocabulary. A child needs to either name a picture (first 4 items, 1 point for a correct answer), or explain the meaning of a word (25 items, giving either 1 or 2 points depending on the correctness of the definition based on the test guidelines). A maximum score for the task was 54 points.

Procedure

The first data collection (t1) took place in spring 2019. Children came in groups of 8–10 students to the data collection site for half a school day. During this "adventure day" the children completed a set of measures individually and in small groups depending on the test format, together with trained research assistants. Small breaks were held between the sessions. The data collection in the third grade (t2) took place in spring 2021. The Covid-19 measures set by the Norwegian government restricted the data collection so that we were not allowed to visit the schools. Instead, one research assistant gave all test instructions online via Teams to one classroom at a time, and the students in the classroom were overlooked by their classroom teacher. The testing was done in two sessions of around 45 minutes each, during one day. The research assistants had a video and audio connections to the classroom. No such technical issues were reported that would have violated the testing situation and validity of the data. Test booklets were delivered to the schools a few days before the data collection took place and collected after the data collection was completed.

The data was coded by trained research assistants, and data from three randomly chosen students per classroom (13%) were double coded by the first author. The correlations between the first and second coding resulted in correlations of sum scores ranging from r = .944–1.00, with coding errors connected to some children having few items in a test with non-matching values. When needed, the test papers compared to the punched values, and the final data matrix corrected accordingly.

Data analysis

First, descriptive statistics of and correlations between all variables were calculated. Second, grouping variables for the MLD status based on sample-based percentiles both in third grade arithmetic fluency and CBM were created. The first grouping variable for arithmetic fluency included MLD25 status and those performing typically (TYP). The second grouping variable for arithmetic fluency included MLD10, LOW and TYP. Similar grouping variables were created for CBM. To answer for RQ1, we tested with a chi-square test what is the overlap of MLD statuses based on arithmetic fluency and curriculum-based mathematics (e.g., children having a status of LOW in both arithmetic fluency and CBM). To answer for RQ2 and RQ3, which first-grade domain-specific and domain-general skills predict MLD status, we conducted two binary logistic regression analyses, one for arithmetic fluency and one for CBM, when having two status groups (MLD25 and TYP), and two multinomial logistic regression analyses when having three status groups (MLD10, LOW and TYP). Jamovi 2.2.2.0 software (The jamovi project, 2021) was used for statistical data analyses.

Results

Descriptive statistics of and correlations between the variables are reported in Table 1. SNMP showed a stronger relation with arithmetic fluency (r = .52) than with CBM (r = .30), whereas counting skills had a moderate relation with both (r = .55 and r = .49, respectively). The associations between domain-general skills and arithmetic fluency were weak with correlations ranging from r = .23 to r = .43, rapid automatized naming showing the strongest relation, and vocabulary the weakest. Regarding the relations with CBM, the strongest association was with nonverbal reasoning, r = . 43, and weakest with rapid automatized naming, r = .21. Multicollinearity was unlikely as all correlations between the predictors were moderate at best.

Means and standard deviations on arithmetic fluency and CBM by each status group are reported in Table 2. As to RQ1, the chi-square test of the crosstabulation of MLD25 and TYP based on arithmetic fluency and CBM was significant, χ^2 (1) = 59.40, p <.001. Sixty-five percent of children were observed as MLD25 in both arithmetic fluency and CBM, while 88% of children as TYP. Similarly, the crosstabulation of three groups, MLD10, LOW, and TYP, based on arithmetic fluency and CBM turned out to be significant, χ^2 (4) = 104.01, p < .001. Sixty-seven percent of the children were observed as MLD10 in both arithmetic fluency and CBM, whereas only 39% of LOW, and 88% of TYP. These results confirmed that a series of separate logistic regression analyses for arithmetic fluency and CBM would be justified.

Predictors of MLD25 Status

Arithmetic fluency (RQ2.1)

A binary logistic regression analysis was performed to ascertain the effects of domain-specific skills (i.e., SNMP and counting skills) and domain-general skills (i.e., nonverbal reasoning, working memory,

Table 1

Descriptive Statistics of and Correlations between the Variables

	1.	2.	3.	4.	5.	6.	7.	8.
1. SNMP t1	—							
2. Counting t1	0.46***	—						
3. Nonverbal reasoning t1	0.25***	0.28***	—					
4. Working memory t1	0.25***	0.41***	0.31***	—				
5. Rapid naming t1	0.42***	0.29***	0.23**	0.33***	—			
6. Vocabulary t1	0.13	0.28***	0.21**	0.21**	0.13	—		
7. Arithmetic fluency t2	0.52***	0.55***	0.28***	0.39***	0.43***	0.23***	—	
8. CBM t2	0.30***	0.49***	0.43***	0.34***	0.33***	0.21**	0.61***	—
M	17.39	20.22	24.18	6.26	10.00	13.45	36.58	34.76
SD	4.13	5.80	5.16	1.74	2.47	3.23	16.90	9.22
Min-Max	7–29	0–29	8–34	0–11	2.09–18.95	2–24	2–89	5–48
Skewness	0.06	-0.89	-0.32	-0.03	0.08	-0.02	0.52	-1.05
Kurtosis	–0.06	0.51	-0.39	0.53	1.32	1.09	0.33	0.89
Cronbach's α	.887	.914	.934	.692	.983	.747	.971	.918

Note. ** $p < .01$, *** $p < .001$. t1 = time point 1 (grade 1), t2 = time point 2 (grade 3), SNMP = Symbolic numerical magnitude processing, CBM = Curriculum-based mathematics*

Table 2

Means and Standard Deviations for Arithmetic Fluency and Curriculum-Based Mathematics (CBM) by Each Status Group

	TYP	MLD25	LOW	MLD10
	M (*SD*) *n*	*M* (*SD*) *n*	*M* (*SD*) *n*	*M* (*SD*) *n*
Arithmetic fluency t2 [a] (max. 90 p.)	43.29 (13.78) 154	16.69 (6.28) 52	21.16 (2.05) 31	10.10 (4.21) 21
CBM t2 [b] (max. 49 p.)	39.14 (4.68) 154	21.81 (6.85) 52	26.60 (1.65) 30	15.27 (5.72) 22

Note. TYP = typically performing (performance >25th percentile), MLD25 = mathematical learning difficulties (performance ≤ 25th percentile), LOW = low-performing (performance between 11–25th percentile), MLD10 = mathematical learning disorder (performance ≤ 10th percentile). [a] The participants in each group are based on their performance on Arithmetic fluency. [b] The participants in each group are based on their performance on CBM. The LOW and MLD10 include the same participants as the MLD25.*

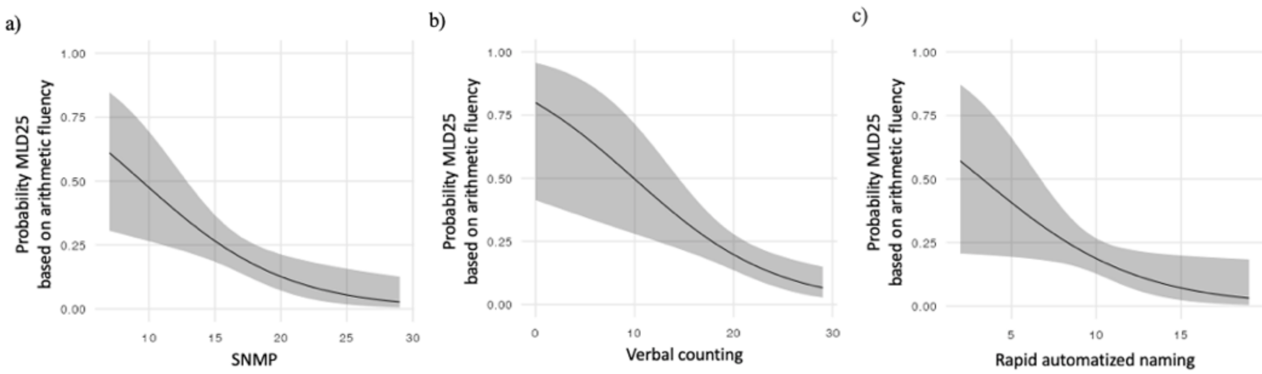
rapid automatized naming, and vocabulary) on the likelihood that participants have MLD25 status. The logistic regression model was statistically significant, χ^2 (6) = 61.69, $p < .001$, and explained 42.0% (Nagelkerke R2) of the variance in MLD status (Table 3). The model correctly classified 83.0% of cases. With a cutoff set at 0.5, the prediction for children with TYP status was more accurate (94.9%) than those with MLD25 (50.0%). Two domain-specific and one domain-general skill predicted the MLD25 status: SNMP (B = -.18, p = .005, odds ratio = .83), counting skills (B = -.14, p = .001, odds ratio = .87), and rapid automatized naming (B = -.22, p = .035, odds ratio = .80). Decreasing performance in these three skills was associated with increasing likelihood of MLD25 status. These are illustrated in Figure 1.

Table 3
Logistic Regression Analysis for MLD25 Status on Arithmetic Fluency

						95% CI for OR	
Predictor	*B*	*SE*	*Z*	*p*	OR	Lower	Upper
Intercept	6.69	1.64	4.07	< .001	NA	NA	NA
SNMP t1	-0.18	0.06	-2.84	0.005	0.83	0.73	0.94
Counting t1	-0.14	0.04	-3.20	0.001	0.87	0.80	0.95
Nonverbal reasoning t1	0.02	0.04	0.46	0.643	1.02	0.94	1.11
Working memory t1	-0.11	0.13	-0.83	0.408	0.90	0.69	1.16
Rapid naming t1	-0.22	0.10	-2.11	0.035	0.80	0.66	0.98
Vocabulary t1	0.02	0.07	0.30	0.767	1.02	0.89	1.17
Model fit measures	Deviance	AIC	R^2N	χ^2	df	*p*	
	149.54	163.54	0.42	61.68	6	< .001	

Note. Estimates represent the log odds of MLD25 vs. TYP (reference group). MLD25 = mathematical learning difficulties (performance ≤ 25th percentile), TYP = typically performing (performance >25th percentile). SNMP = symbolic numerical magnitude processing, OR = odds ratio, CI = confidence interval, AIC = Akaike information criterion, R^2N = Nagelkerke's R^2.

Figure 1
Predicted Probability with 95% Confidence Interval for a Status of MLD25 in Arithmetic Fluency versus a) Symbolic Numerical Magnitude Processing (SNMP), b) Verbal Counting, and c) Rapid Automatized Naming



Curriculum-based mathematics (RQ2.2)

A similar binary logistic regression analysis was conducted for CBM as for arithmetic fluency. The logistic regression model showed to be statistically significant, χ^2 (6) = 34.89, *p* < .001, and explained 26.0% of the variance in MLD25 status (Table 4). The model correctly classified 79.0% of cases. With a cutoff set at 0.5, the prediction for children with TYP status was more accurate (94.2%) than those with MLD25 (34.8%). One domain-specific and one domain-general skill predicted the MLD25 status: counting skills (*B* = -.09, *p* = .026, odds ratio = .92) and nonverbal reasoning (B = -.12, p = .003, odds ratio = .89). Decreasing performance in these two skills was associated with a higher probability of MLD25 status, as illustrated in Figure 2.

Predictors of MLD10 and LOW Status

Arithmetic fluency (RQ3.1)

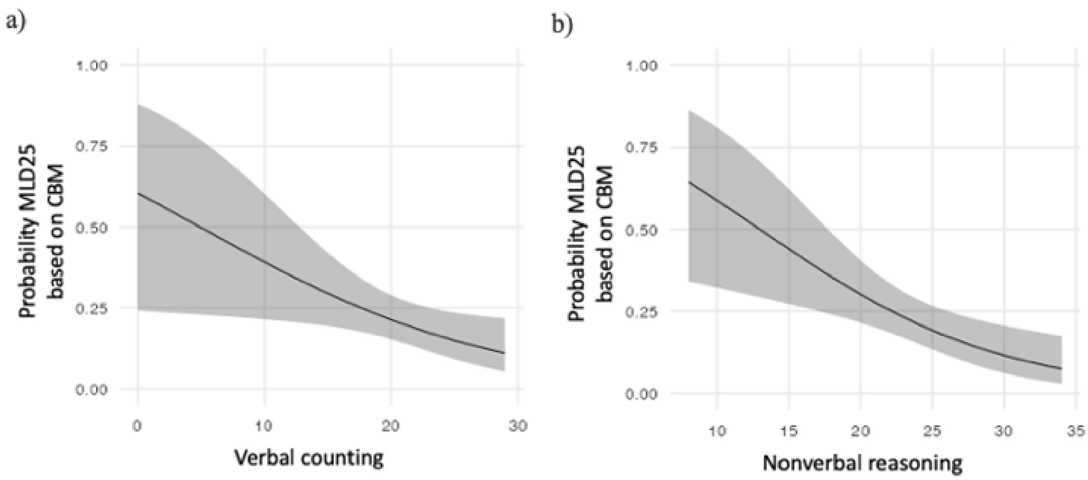
A multinomial logistic regression analysis was run with a dependent variable of status consisting of three status groups (TYP, LOW, MLD10). The model was statistically significant, χ^2 (12) = 65.01, *p* < .001, and explained 28.0% of the variance in status (Table 5). When MLD10 was compared to TYP, the results were similar to the MLD25 status; SNMP, counting skills, and rapid automatized naming, predicted MLD10 status (SNMP: *B* = -.22, *p* = .019, odds ratio = .81; counting skills: *B* = -.16, *p* = .008, odds ratio = .85; and rapid automatized naming: *B* = -.28, *p* = .038, odds ratio = .75). When LOW was compared to TYP, only domain-specific skills, SNMP (*B* = -.17, *p* = .021, odds ratio = .85) and counting

Table 4
Logistic Regression Analysis for MLD25 Status on Curriculum-Based Mathematics

						95% CI for OR	
Predictor	*B*	*SE*	*Z*	*p*	OR	Lower	Upper
Intercept	4.67	1.31	3.57	< .001	NA	NA	NA
SNMP t1	0.02	0.06	0.36	0.715	1.02	0.91	1.14
Counting t1	-0.09	0.04	-2.23	0.026	0.92	0.85	0.99
Nonverbal reasoning t1	-0.12	0.04	-2.94	0.003	0.89	0.82	0.96
Working memory t1	-0.13	0.12	-1.06	0.287	0.88	0.69	1.12
Rapid naming t1	-0.11	0.09	-1.21	0.225	0.89	0.75	1.07
Vocabulary t1	0.02	0.06	0.27	0.788	1.02	0.90	1.15
Model fit measures	Deviance	AIC	R^2N	χ^2	df	*p*	
	172.05	186.05	0.26	34.89	6	< .001	

Note. Estimates represent the log odds of MLD25 vs. TYP (reference group). MLD25 = mathematical learning difficulties (performance ≤ 25th percentile), TYP = typically performing (performance > 25th percentile). SNMP = symbolic numerical magnitude processing, OR = odds ratio, CI = confidence interval, AIC = Akaike information criterion, R^2N = Nagelkerke's R^2.

Figure 2
Predicted Probability with 95% Confidence Interval for a Status of MLD25 in Curriculum-Based Mathematics versus a) Verbal Counting and b) Nonverbal Reasoning



skills (B = -.13, p = .009, odds ratio = .88) were significant predictors for LOW status. None of the predictors were significant when comparing MLD10 with LOW. Figure 3 illustrates how decreasing performance in SNMP and counting skills are associated with an increased likelihood of MLD10 and LOW status, and similarly for rapid automatized naming for MLD10 status.

Curriculum-based mathematics (RQ3.2)

A similar multinomial logistic regression analysis was done for CBM. The model was statistically significant, χ^2 (12) = 47.12, p < .001, and explained 21% of the variance in status (Table 6). When MLD10 was compared to TYP, the same two factors that had predicted MLD25 predicted also MLD10 status: counting skills (B = -.15, p = .006, odds ratio = .86) and nonverbal reasoning (B = -.11, p = .046, odds ratio = .89). In addition, rapid automatized naming (B = -.28, p = .033, odds ratio = .76) predicted MLD10 status. When LOW was compared to TYP, again, nonverbal reasoning predicted the status (B = -.13, p = .009, odds ratio = .88), but counting skills was no longer a significant predictor. Instead, working memory predicted LOW status (B = -.33, p = .035, odds ratio = .72). These differences were also visible when comparing MLD10 versus LOW. Decreasing working memory skills was associated with a higher probability of LOW status than MLD10, and vice versa for rapid automatized naming. Figure 4 illustrates the predicted probabilities for the status of TYP, LOW, and MLD10 versus counting skills, nonverbal reasoning, working memory and rapid automatized naming.

Table 5

Multinomial Logistic Regression Analysis for MLD10 and LOW Status on Arithmetic Fluency

						95% CI for OR	
Predictor	*B*	*SE*	*Z*	*p*	OR	Lower	Upper
MLD10–TYP							
Intercept	7.30	2.12	3.44	< .001	NA	NA	NA
SNMP t1	-0.22	0.09	-2.35	0.019	0.81	0.67	0.96
Counting t1	-0.16	0.06	-2.65	0.008	0.85	0.76	0.96
Nonverbal reasoning t1	-0.01	0.06	-0.16	0.876	0.99	0.88	1.11
Working memory t1	-0.09	0.19	-0.49	0.622	0.91	0.63	1.32
Rapid naming t1	-0.28	0.14	-2.07	0.038	0.75	0.58	0.98
Vocabulary t1	0.05	0.10	0.49	0.625	1.05	0.86	1.28
LOW–TYP							
Intercept	5.34	1.78	2.99	0.003	NA	NA	NA
SNMP t1	-0.17	0.07	-2.31	0.021	0.85	0.74	0.98
Counting t1	-0.13	0.05	-2.63	0.009	0.88	0.80	0.97
Nonverbal reasoning t1	0.03	0.05	0.72	0.471	1.04	0.94	1.14
Working memory t1	-0.12	0.15	-0.80	0.421	0.89	0.66	1.19
Rapid naming t1	-0.18	0.11	-1.62	0.105	0.83	0.66	1.04
Vocabulary t1	0.01	0.08	0.10	0.923	1.01	0.86	1.18
MLD10–LOW							
Intercept	1.97	1.98	0.99	0.320	NA	NA	NA
SNMP t1	-0.05	0.10	-0.52	0.604	0.95	0.79	1.15
Counting t1	-0.03	0.06	-0.52	0.602	0.97	0.86	1.09
Nonverbal reasoning t1	-0.04	0.07	-0.67	0.501	0.96	0.84	1.09
Working memory t1	0.03	0.20	0.12	0.901	1.03	0.69	1.53
Rapid naming t1	-0.10	0.13	-0.73	0.468	0.91	0.70	1.18
Vocabulary t1	0.04	0.10	0.40	0.689	1.04	0.85	1.27
Model fit measures	Deviance	AIC	R^2N	χ^2	df	*p*	
	210.64	238.65	0.28	65.01	12	< .001	

Note. Estimates represent the log odds of MLD10 vs. TYP (reference group), LOW vs. TYP (reference group), and MLD10 vs. LOW (reference group). MLD10 = mathematical learning disorder (performance ≤ 10th percentile), TYP = typically performing (performance >25th percentile), LOW = low-performing (performance between 11–25th percentile). SNMP = symbolic numerical magnitude processing, OR = odds ratio, CI = confidence interval, AIC = Akaike information criterion, R^2N = Nagelkerke's R^2.

Figure 3

Predicted Probability for a Status of MLD10, LOW and TYP in Arithmetic Fluency versus a) Symbolic Numerical Magnitude Processing (SNMP), b) Verbal Counting, and c) Rapid Automatized Naming

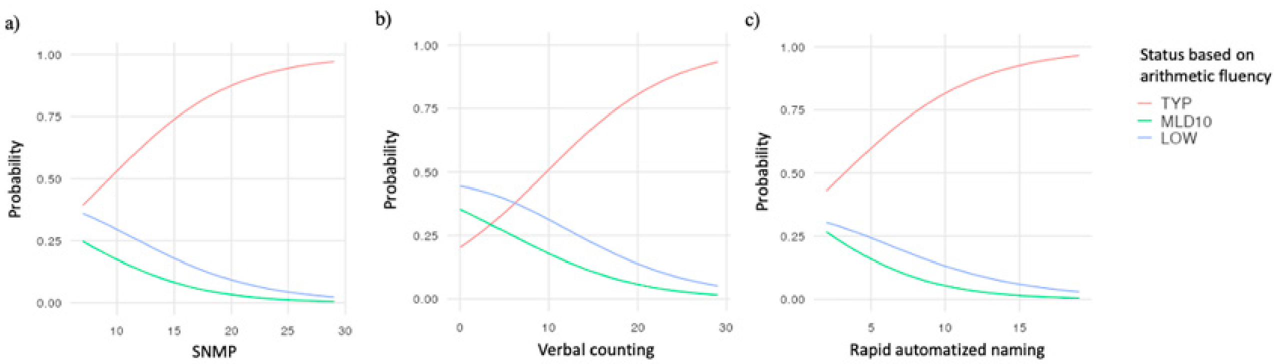


Table 6

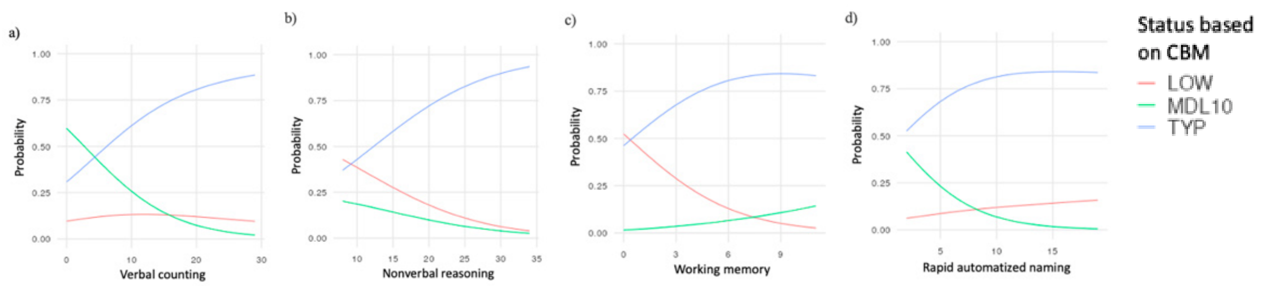
Multinomial Logistic Regression Analysis for MLD10 and LOW Status on Curriculum-Based Mathematics

						95% CI for OR	
Predictor	*B*	*SE*	*Z*	*p*	OR	Lower	Upper
MLD10–TYP							
Intercept	4.28	1.72	2.49	0.013	NA	NA	NA
SNMP t1	0.02	0.08	0.20	0.840	1.02	0.87	1.19
Counting t1	-0.15	0.06	-2.77	0.006	0.86	0.77	0.96
Nonverbal reasoning t1	-0.11	0.06	-2.00	0.046	0.89	0.80	1.00
Working memory t1	0.15	0.19	0.79	0.431	1.16	0.80	1.69
Rapid naming t1	-0.28	0.13	-2.13	0.033	0.76	0.59	0.98
Vocabulary t1	0.05	0.09	0.60	0.551	1.06	0.88	1.26
LOW–TYP							
Intercept	3.49	1.58	2.20	0.028	NA	NA	NA
SNMP t1	0.02	0.07	0.36	0.719	1.03	0.90	1.17
Counting t1	-0.04	0.05	-0.78	0.435	0.96	0.88	1.06
Nonverbal reasoning t1	-0.13	0.05	-2.60	0.009	0.88	0.80	0.97
Working memory t1	-0.33	0.15	-2.11	0.035	0.72	0.53	0.98
Rapid naming t1	0.03	0.12	0.25	0.806	1.03	0.82	1.29
Vocabulary t1	-0.02	0.08	-0.23	0.818	0.98	0.85	1.14
MLD10–LOW							
Intercept	0.79	1.99	0.40	0.690	NA	NA	NA
SNMP t1	-0.01	0.09	-0.09	0.927	0.99	0.82	1.19
Counting t1	-0.12	0.06	-1.82	0.069	0.89	0.79	1.01
Nonverbal reasoning t1	0.01	0.07	0.19	0.849	1.01	0.89	1.16
Working memory t1	0.48	0.23	2.09	0.036	1.61	1.03	2.52
Rapid naming t1	-0.31	0.16	-1.98	0.048	0.74	0.54	1.00
Vocabulary t1	0.07	0.10	0.69	0.490	1.07	0.88	1.32
Model fit measures	Deviance	AIC	R^2N	χ^2	df	*p*	
	222.80	250.80	0.21	47.12	12	< .001	

Note. Estimates represent the log odds of MLD10 vs. TYP (reference group), LOW vs. TYP (reference group), and MLD10 vs. LOW (reference group). MLD10 = mathematical learning disorder (performance ≤ 10th percentile), TYP = typically performing (performance >25th percentile), LOW = low-performing (performance between 11–25th percentile). SNMP = symbolic numerical magnitude processing, OR = odds ratio, CI = confidence interval, AIC = Akaike information criterion, R^2N = Nagelkerke's R^2.

Figure 4

Predicted Probability for a Status of MLD10, LOW and TYP in Curriculum-Based Mathematics versus a) Verbal Counting b) Nonverbal Reasoning, c) Working Memory, and d) Rapid Automatized Naming



Discussion

This study investigated how domain-specific (i.e., SNMP and counting skills) and domain-general skills (i.e., nonverbal reasoning, working memory, rapid automatized naming, and vocabulary) measured in the first grade predict MLD status among third graders. Several studies have found both domain-specific (e.g., Cañizares et al., 2012; Desoete et al., 2012; Hassinger-Das et al., 2014; Landerl et al., 2004) and domain-general skills (e.g., David, 2012; Huijsmans et al., 2022; Mazzocco & Grimm, 2013) to be negatively associated with MLD. However, relatively few studies have considered whether these relations are dependent on how MLD has been defined and operationalised in terms of the severity of MLD and measures of mathematics performance. The novelty of this study is that it took into consideration both the different cut-off criteria and the measures of mathematics performance (i.e., arithmetic fluency and curriculum-based mathematics) in defining the MLD status, and included several early domain-specific and domain-general skills as predictors of MLD. Our findings suggest that both different cut-off criteria and mathematics measures used for the definition of the MLD status are important to acknowledge, as these led to relatively significant variation in which students were identified as having MLD and which domain-specific and domain-general factors contributed to the MLD status.

Prior research has used various different mathematics measures for the identification of students with MLD. Typically, arithmetic fluency (Chow et al., 2021; Koponen, Aro, et al., 2018) and broader mathematics performance tests (Jordan et al., 2002) have been applied. Little is known whether these different measures identify the same participants under the same MLD status. Therefore, in our study, we first examined the overlap of MLD statuses (MLD25, MLD10 and LOW) based on arithmetic fluency and CBM (RQ1). For the MLD25 status, the overlap was 65 %, while for the MLD10 status, the share was 67 %, and for LOW, only 39 %. These results show that the use of only one type of the mathematics measure would have missed a number of children struggling with either arithmetic fluency or CBM. Consequently, using a measure reflecting one area of mathematics to define MLD may lead to an exclusion of students with difficulties in another equally relevant area of mathematics. It is thus important to consider which mathematics measures to use for the identification of MLD, and whether to rely on one or multiple measures. As we are still lacking a globally applicable diagnostic measure of MLD, we would encourage researchers to carefully report both the cut-off criteria and mathematics measures used for the identification of MLD and MLD status (e.g., LOW, MLD10) for better comparability of research findings.

Confirming that both arithmetic fluency and CBM measures are important to consider, we focused next on how domain-specific and domain-general skills predict different MLD status based on arithmetic fluency or curriculum-based mathematics.

Predictors of MLD based on Arithmetic Fluency

As hypothesized (H2.1), MLD25 status based on arithmetic fluency was predicted by SNMP, verbal counting skills and rapid automatized naming. The lower was the first-grade performance in these skills, the higher was the probability of showing weak arithmetic fluency (MLD25) in the third grade. A similar pattern of predictions was found when the MLD25 group was divided into MLD10 and LOW. As expected (H3.1), SNMP and counting skills predicted both statuses, but rapid automatized naming predicted only MLD10. That is, SNMP and counting skills predicted the status of MLD based on arithmetic fluency independent of the severity level of MLD. Weakness in early rapid automatized naming, instead, seemed to be more strongly associated with MLD10.

Prior research has shown the importance of early mathematical skills for later arithmetic and mathematics performance (Aunio & Niemivirta, 2010; ten Braak et al., 2022), which our findings support. The role of (S)NMP in mathematics learning and MLD has been under debate due to mixed findings (Cañizares et al., 2012; De Smedt et al., 2013; De Smedt & Gilmore, 2011; Desoete et al., 2012; Mammarella et al., 2021). Our findings give further support to SNMP being an important factor for later mathematics performance, as we found that SNMP measured at grade 1 was a significant predictor of MLD status based on arithmetic fluency at grade 3. The task measuring SNMP involves recognition of number symbols (1-digit numbers) and understanding their related magnitude (Brankaer et al., 2017). It could be that this type of basic symbolic magnitude processing in the beginning school is relevant especially for arithmetic learning at school (De Smedt et al., 2013; Nosworthy et al., 2013), and thus a good predictor of MLD based on arithmetic fluency. Also, verbal counting skills (i.e., knowledge of number sequences) was found to predict MLD status based on arithmetic fluency. Prior research has shown that verbal counting skills are important for learning basic addition and subtraction skills, and used also as a strategy for solving unknown addition and subtraction facts (Koponen et al., 2019; Ostad, 1998). Therefore, difficulties in early verbal counting skills may slow down the learning of arithmetic facts. For these children, solving addition and subtraction facts may become more error-prone due to making mistakes in number sequences, and continuously getting incorrect answers between the fact and the answer may thus interrupt memorizing the facts fluently.

Rapid automatized naming was found to be the only domain-general predictor of MLD status based on arithmetic fluency, which is in line with prior research (Donker et al., 2016; Mazzocco & Grimm, 2013). While we found non-alphanumeric rapid automatized naming to predict MLD25 and MLD10 status, Donker et al. (2016) similarly reported students with MLD25 to have weaker non-alphanumeric rapid automatized naming skills. Further, Mazzocco and Grimm (2013) found students with MLD10 to have persistent weakness in rapid automatized naming of colors compared to their peers, and also students with LOW status to show slight delay in their development of rapid naming of colors. As to why rapid automatized naming and arithmetic fact retrieval are related, in both tasks children need to access quickly and retrieve phonological representations from long-term memory (e.g., "blue" and "seven"). The role of early non-alphanumeric rapid automatized naming skills in later MLD based on arithmetic fluency seems to be important to acknowledge.

Based on our findings, both weak SNMP and counting skills could be considered as risk factors for later difficulties in arithmetic fluency, independent of the level of severity, while rapid automatized naming seems to be specifically associated with MLD10 students' arithmetic fluency. As a practical implication for early schooling, SNMP and verbal counting skills should be regularly screened in classrooms (see e.g., Brankaer et al., 2017; Nosworthy et al., 2013; Salminen & Koponen, 2011). Those who struggle in comparing 1-digit numbers or in reciting number sequences forwards and backwards, should be provided with relevant intensified pedagogical support (i.e., intervention) as early as possible (see e.g., Ramani et al., 2017). Even if the role of early rapid automatized naming skills in MLD is important to acknowledge, training of domain-general skills, with many examples from working memory training research, has shown rather weak far transfer effects on mathematics performance (Melby-Lervåg & Hulme, 2013). However, recently Pecini et al. (2019) showed the training of rapid automatized naming using specific software to be effective in ameliorating reading accuracy and speed. Until we have solid evidence of the effectiveness of training rapid automatized naming and its transfer on improved mathematical skills, it might be more useful to focus on training mathematical skills as a preventive step with children identified with weakness in early rapid automatized naming skills.

Predictors of MLD based on Curriculum-Based Mathematics

Our findings on the predictions of MLD based on CBM differed from those of MLD based on arithmetic fluency. Here, the domain-general skills seemed to be better predictors than domain-specific skills, especially in relation to MLD10 and LOW status. Because the tasks in CBM had more variety and complexity in their mathematics content and procedures, we expected counting skills (Hassinger-Das et al., 2014), nonverbal reasoning (Engle, 2018; Huijsmans et al., 2022), working memory (David, 2012), and vocabulary (Chow et al., 2021) to predict MLD25 status. In accordance with our hypothesis (H2.2), counting skills and nonverbal reasoning, but not working memory, were found to predict MLD25 status based on CBM. When MLD25 was divided into two, counting skills only predicted the MLD10 status, but not LOW, as we would have expected (H2.3). In fact, no domain-specific skills predicted LOW status based on CBM. Regarding domain-general skills, in contrast to our hypothesis, nonverbal reasoning and rapid automatized naming, but not working memory, predicted MLD10 status based on CBM. Instead, working memory together with nonverbal reasoning predicted LOW status.

As elaborated above, early counting skills have shown to be associated with later mathematics performance (ten Braak et al., 2022). Concerning MLD status, our results revealed that verbal counting skills predicted especially MLD10 status based on CBM. Taken together, these findings imply that verbal counting skills are good at predicting MLD10 status independent of the mathematics measure used for identification. This further puts emphasis on supporting children's early counting skills in early schooling as a preventive step for later severe MLD.

Nonverbal reasoning was found to predict MLD status based on CBM, independent of the severity level. CBM test measured a broad range of mathematical subskills with different types of tasks (e.g., "Lisa's little finger is 4 g/kg/cm/m long"; "One apple costs 3 krones, 4 apples cost __ krones.", simple word problems for fractions, and addition and subtraction algorithms). Solving these tasks thus required making logical decisions and proceeding systematically in the task (i.e., nonverbal reasoning) (Engle, 2018), which differs from solving simple arithmetic facts. This result also resonates well with Hujsmans' et al. (2022) findings showing students with MLD to perform weaker in nonverbal reasoning compared to their peers without MLD.

MLD10 and LOW based on CBM were separated by the domain-general predictors of rapid automatized naming and working memory (i.e., central executive functioning). Overall, rapid automatized naming turned out to be an important predictor of MLD10 status, as it predicted the status based on both arithmetic fluency and CBM. Previously, rapid automatized naming has been found to be related to broader mathematics performance as well, although not as strongly as to arithmetic fluency (Koponen et al., 2017). Because the tasks in mathematics performance tests typically involve more processes than quick retrieval only, the relationship of rapid automatized naming with broader mathematics performance are weaker than with arithmetic fluency. This was also evident in our study (r = .43 with arithmetic fluency, and r = .33 with CBM).

Interestingly, working memory, and more specifically, central executive, predicted only LOW status based on CBM, although prior research has found its role to be significant in mathematics learning (Friso-van den Bos et al., 2013) and MLD. Students with MLD often have a working memory deficit (Andersson & Lyxell, 2007; David, 2012; Passolunghi & Mammarella, 2010). Concerning the MLD status here, our results partly reflect the findings by Huijsman et al. (2022), who found students identified as LOW to be weaker than MLD10 in the visual component of working memory. Even if we used a generally recognized backwards digit span as a measure of central executive in our study, it might be that the central executive measured this way captured only part of the construct, and a broader measure would have been needed to retain its predictive power and thus obtain a similar effect on MLD as in previous studies.

Limitations and Future Directions

The current study has some limitations that need to be noted. First, although we included many domain-specific and domain-general skills based on prior research, we may have missed some other important factors as predictors, such as non-symbolic numerical magnitude processing, subitizing, or object counting as domain-specific skills, or inclusion of other components of working memory (i.e., phonological loop and visuo-spatial working memory) and language (e.g., receptive vocabulary or syntax). Their role as a predictor of MLD would be of future interest to explore.

Concerning the severity criteria for the MLD status, we based our grouping of children on sample-based percentiles both in arithmetic fluency and CBM. It might be argued that it would have been better to use mathematics tests with norms for identifying students with MLD. At the time of the study, neither standardized broad mathematics performance tests nor combined tests of addition and subtraction facts were available for this age group in Norway. Therefore, we developed a new CBM test, and applied sample-based percentiles in both CBM and arithmetic fluency (i.e., a combination of addition and subtraction fluency subtests) for the identification of MLD. The lack of global standardized mathematics measures to be used in MLD studies is problematic in terms of the comparability of results, and should be addressed in future research on MLD.

Note, that because of practical reasons, our sample was restricted to only include children from the Oslo region in Norway, due to which caution should be exercised in generalizing the findings to other contexts. Also, the Covid-19 pandemic complicated the final stages of the data collection due to which one school withdrew from the study and the data collection in spring 2021 needed to be organized online. However, there are no indications of this causing any bias in our data.

Conclusions

Our findings suggest that both the different cut-off criteria and mathematics measures used for defining and operationalizing the MLD status are important to acknowledge in studies, as these may lead to relatively significant variation in which students are identified as having MLD and which factors contribute to the MLD status. In relative terms, domain-specific skills appear to be more predictive when the MLD status is based on arithmetic fluency, while domain-general skills seem more influential when the MLD status is based on CBM. Counting skills and rapid automatized naming, instead, appear to be robust predictors of MLD status regardless of the mathematics measure used. As a practical implication for the prevention of MLD, we advocate focusing on screening children's SNMP and verbal counting skills in early grades, and providing appropriate intervention in these for those in need of educational support.

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Zero – an Uncommon Number: Preschoolers' Conceptual Understanding of Zero

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Abstract

The conceptual development of natural number in preschoolers is well-researched. However, less is known about the conceptual development of zero. Recent studies suggest that children develop an understanding of zero after learning to count. It remains unclear, when a conceptual understanding of "zero" as number word for an empty set emerges. This paper integrates numerical and language theories about how, where and when the concept of zero is formed and is integrated into the class of natural numbers. The counting skills of 107 preschoolers were assessed for the number range between zero and eight as well as for their ordinal understanding of zero. The results show that compared to the natural numbers, zero was substantially more difficult. Children are able to list zero in a number word sequence (0, 1, 2, 3 .... or 3, 2, 1, 0), but were unable to describe a set as having zero numbers. This latter conception contradicts findings regarding natural numbers, in that an empty set is counter intuitive. Zero could be correctly placed when consecutive order was required, but addition and subtraction by counting was more difficult. The results suggest that the conceptual development of zero differs qualitatively from the natural numbers. Based on the results, the ordinal understanding of zero as a predecessor to one, together with its matching linguistic concepts is proposed to be the key to the conceptual development of zero.

Keywords:

Numerical Cognition, Zero, Ordinal Representation, Early Numeracy, Conceptual Development

Introduction

Much is known and researched about how children learn the concept of natural number, but when it comes to "zero", there is much not known. We generally use words to describe the nothingness or emptiness in everyday relations and we naturally talk about the lack of something, for example, "There are two apples, but there are no bananas." We reject objects, things or conditions and therefore form relations of nothing. All this seems facile since even children as young as two utter sentences like, "there are no cookies on my plate". But talking about "zero" and referring to it mathematically as an empty set seems to be much more complex. Going back in history it can be seen that to equip zero with an unique symbol and to integrate this into the ordinal sequence of natural number was a long journey that spanned many centuries, and followed different paths. The Babylonian system was one of the earliest to include place value when recording amounts. They had a dot as placeholder sign to indicate, where a specific place value was zero (e.g. in 2022, where there are zero hundreds). However, the idea that a sign could represent zero items was revolutionary, when it came up in ancient India. Bottazzini (2021) states that about the year 628 AD, the function of zero transformed from merely being a placeholder for an empty position in the notation of a number to a natural number with the consequent properties of a natural number. In Europe it was not until Fibonacci's Liber Abaci in 1202 (Sigler, 2002) that "zero" was broadly accepted as part of the number system (Ifrah, 1998).

So, what appears to be difficult is not to talk about nothing, but to mathematically frame a concept for zero and integrate this into the class of natural number. As many as 15% of preservice elementary school teachers do not refer to zero as a natural number (Krajcsi et al., 2021). It is clear that there is a lack of understanding how children develop the concept "zero" as both an "empty set" and as a "placeholder". All we know for sure is that the concept of "zero" seems to be harder to learn than the concepts of "one, two, three…". In the present study we will briefly summarize what is known about the understanding of zero and will frame the problem in theoretical terms specifying the lexical concepts of natural number which then provides the basis for zero as an abstract numerical concept. We will then present data showing the developmental hierarchy of natural number and specify how the progressive understanding of "zero" develops. Finally, we will suggest an outline of how "zero" is handled when it comes to the ordinal dimension of the number line.

Development of Natural Number

From the very beginning children encounter numbers, values and sets of things. Learning to speak means to build references between objects or actions and the corresponding vocabulary. While starting with mostly content words, productive vocabulary is from the start used to describe relationships between objects. At two years of age children no longer seem to have the need to refer to each object in singular form but start to refer to sets of similar things using natural quantifiers (Barner et al., 2007). What is remarkable here is that this ability to use natural quantifiers, forms the foundation to engage verbally with the world of numerical relationships. Soon after, the first concrete denomination of a set of two occurs. Children now refer to two entities as being exactly two whereas earlier they had used a natural quantifier like "many" instead. Using exact number words to describe their surroundings, children refer to lexical concepts which are concrete and abstract at the same time. Whereas the "twoness" of something is a unique, distinct, and therefore concrete feature, it can differ in shape, color, form and size (Wiese, 2007), which gives it a degree of abstraction. Unlike for example "yellow" which refers to the characteristic of the object, number words will always refer to the relation the objects hold with each other. So "two" as a lexical concept will almost always have a different referent while the numerical value forms the linking, stable element.

An often-cited theory identifies innate knowledge of number and magnitude as well as language features as underling this developmental process. Innate knowledge of number and magnitude is described through two evolutionary old systems which together form the core systems (Dehaene, 1999). The approximate number system, being the first of the two, holds information of magnitude. It represents a physical magnitude cognitively by a roughly proportional cardinal value. It is stable over different dimensions like brightness, loudness, and temporal duration, and could be shown in children as young as six months. It underlies Weber`s law, meaning it is increasingly harder to discriminate the absolute distance of two entities of greater magnitudes (Sarnecka & Carey, 2006). Discrimination starts with a ratio of 1:2 and can sharpen up to 9:10 (Halberda & Feigenson, 2008).

The second core system processes mental representations up to a limit of three. With the object tracking system, discrete objects are stored in individual object-files holding one up to three elements. Being nonverbal, object files are compared as being equal or unequal to their match in the world. It has been shown that children use these files to distinguish entities according to quantity (Wynn, 1992) and that it does not work for entities higher than four (Feigenson et al., 2002). Going onwards children rely on counting to form concepts of natural number. Counting principles, introduced by Gelman and Gallistel (1978), form a hierarchy of how counting helps children to get a better insight into ordinal and cardinal aspects of natural number. One of the principles states that the last number word in a counting process represents the magnitude. This principle implies knowledge that going onward in the number line means increasing magnitude.

Language has been presented by Carey (2009) as a third indispensable system for the development of natural number. As stated briefly in the introduction, language has the power to discriminate between singular and plural. Moreover, language, or more precisely the class of number words, forms the scaffolding to which numerical information is attached. Thereby, surface concepts of natural number are formed and will then be specified throughout development (Hartmann & Fritz, 2021). The number word sequence up to ten is learned and memorized in stable order shortly after the second birthday. Being just a string of words at this point in time, it does not hold very deep numerical knowledge, but each number name provides a hook, for the specific lexical concepts of natural number (Negen & Sarnecka, 2012). Ordinal and cardinal aspects of natural number are over a long period sequentially integrated into the string of words, number names, originally learnt.

To build these concepts children make use of the conceptual function bootstrapping, a term coined by Carey (2009). Bootstrapping describes the necessity to combine all three systems and construct completely new concepts. To actually possess understanding of all features means more than simply match a number word to its corresponding individual object file. In fact, Le Corre and Carey (2007) describe the laborious and slow process children go through to map the number words one up to four to the corresponding mental representations. Forming these new lexical concepts takes about one year to develop. And even though cardinal development of natural number seems to move faster after constructing the concept four, the precise semantic mapping of a number word larger than four to the corresponding magnitude still needs about six more months to develop (Le Corre and Carey, 2007). Not until then, children will answer with an approximately close number word when presented with a random magnitude. Prior to this development, their answers are arbitrary. It almost seems as if the approximate number system needs to sharpen, that means to map closely matching sets and number words automatically.

These mappings of a number word to its corresponding magnitude sequentially fills the sequence of number words with numerical information. Based on counting and the stable order of the number word sequence a change in the representation of numbers takes place and numbers become associated with the order of successive quantities. In this mental representation, the successive number words align gradually to increasing quantities. A kind of "mental number line" is constructed this way and forms an ordinal representation (Fritz et al., 2018; Le Corre, 2014). With this knowledge, numbers can be compared to each other according to their position on the number word line, ("which number is bigger 7 or 8"?) and children are able to identify preceding and succeeding numbers ("which number comes before 3, and after 3"?).

The representation of the mental number line allows children to solve basic addition and subtraction tasks by counting. "The rabbit has two carrots and gets two more. How many does it have now?" Tasks like these can be completed by counting forward, always beginning from one and identifying the name of the number they found out as a result. Ordinal concepts do not yet include knowledge of cardinality.

Development of "Zero"

In none of these findings and principles, discussed above can zero be integrated. There seems to be no matching object file for "zero" in the object tracking system, its "magnitude" cannot be embodied by the approximate number system and it does not play any part in the early mental number line. In addition, the mathematical term, "zero", is not in the common vocabulary of young infancy. There are just very few and often contradictory findings about the understanding of zero. However, all of the studies prove pointers to the actual problem of understanding zero.

Wellman and Miller (1986) worked with Arabic notation and verbal count items ranging from 0 to 5. They found a delay in the use of zero compared to the rest of the natural numbers. They stated that children could name the symbol "0" around the fourth birthday and that children six years of age could describe zero as being the smallest number and could compare numbers. The findings of Bialystok and Codd (2000) contradicted these observations. They worked with a "Give-me" task to investigate children's knowledge of natural number including zero. They conclude, that preschoolers understand the concept of zero and can solve "give-zero" tasks. It is important to note here, that they did not ask to "give zero cookies" but rather to "give no cookies". Merrit and Brannon (2013) state that zero is handled differently by children and might not even be considered to be a number since it is not part of the counting list. And even though children could state that zero is smaller than one they did not naturally categorize it to be a number (Krajcsi et al., 2021).

One main problem seems to derive from linguistics, more precisely, the vocabulary. Spoken language usually does not refer to empty sets as being zero but uses a variety of different words or phrases to describe the characteristic of an empty set. Zero is characterized and referred to as no apples, nothing to eat, empty glass, vacant chairs, blank spaces. One problem in addressing zero might therefore be its low frequency use and the different realizations in spoken language. In contrast to natural numbers which in everyday life is referred to by precise number words, "zero" is usually referred to semantically indirect references, e.g. "no", "empty" or "nothing".

Nonetheless, young children are capable of working with empty sets in everyday life. To draw an analogy, all these "empty-set-words" do hold numerical content the way natural quantifiers do. But unlike other natural quantifiers they do not naturally find the corresponding mathematical denomination. So, the number word "zero" might have the difficulty of being doubly abstract. During development there is

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not one prominent number word with which children can match the distinctness of emptiness. The second abstract feature is, that when referring to "no entities", these entities do not have a match in the real word. There are no apples, sheep, cars, marbles or whatever to be seen. From a linguistic perspective, children need to create a concept for the lack of something instead of a concept of the magnitude of something.

In summary, the origin of the more difficult and later emerging understanding of zero might be explained in the double abstraction of linguistically no matching referent in the real world and not one specific number word combined with numerically no anchor point in the numerical core systems since neither of the core systems are laid-out to represent the lack of something but rather to present magnitudes. This explanation does not preclude that comparisons of empty-sets and magnitudes are possible. These are possible even in very early infancy since everyday language provides vocabulary for the lack of something.

Research questions

Common theories postulate a hierarchy in the development of precise concepts of natural numbers: Children develop the concept of "one" before they develop the concept of "two", and "three" is developed after "two". A growing body of studies bolster this assumption based on empirical data (e.g. Le Corre et al., 2006; Sarnecka & Lee, 2009; Negen & Sarnecka, 2012). Obviously, children do not develop a concept of "zero" before they have developed the concept of "one". Quite the opposite - since the acquisition of the concept of zero is much more abstract, recent studies suggest that children do not acquire the concept of "zero" until they have learned some number words, at least the number words 1 – 4, indicating that they are cardinal principle knowers (Krajcsi et al., 2021; Pixner et al., 2018).

Based on the concept of cardinal principle knowledge, an initial understanding of the relations between numbers develops. Children start to construct an ordinal number line, in which numbers are aligned as gradually increasing quantities. Empirical evidence shows that such an ordinal understanding of the natural numbers implies an understanding of the meaning of number words greater than 4 (Fritz et al., 2018; Le Corre, 2014). But even if, according to the findings of Krajcsi et al., children perceive 0 as smaller than 1, the question of what previous knowledge is required in order to integrate 0 into the mental number line of increasing quantities has not been answered.

These findings raise two main research questions that we aim to address in this study:

1. When in the process of acquiring the meaning of the natural numbers one to eight does the understanding of the natural number 0 develop? Given the high level of abstraction of the number zero, it is expected that the understanding of the number zero will emerge only when the children have mastered at least the meaning of the numbers one to four.

2. Does children's development of an ordinal concept of "zero" require a cardinal concept of "zero"? Does the integration of the number zero into the ordinal number line only happen after the meaning of the number words zero to eight have been grasped - in other words, does the integration of 0 into this list require an understanding of the natural numbers zero to eight?

Methods

Sample

In this study, a total of *N* = 107 kindergarteners (62 female, 45 male) participated. The children's mean age was M_{age}=57.61 months (SD_{age}=7.88 months), ranging from 44 months to 71 months. 30 children spoke German and an additional language at home, while 7 children did not speak German, but another language at home. Most common foreign home languages were English (*n* = 15), Arabic (*n* = 4), Turkish (*n* = 3), and Polish (*n* = 3). Children were recruited in 11 kindergartens from mostly urban backgrounds. Kindergartens were selected with the aim to represent upper-class (3 kindergartens), middle-class (4 kindergartens), and lower-class (4 kindergartens) backgrounds.

In advance, parents and children were informed about the procedures and aims of the study. Written consent was obtained from the parents beforehand. All national research standards were met during this study. The data collection was done by three experienced graduate university students that were trained by the researchers responsible for the study.

Instruments

Give-N: Children's counting skills were assessed with the Give-N task. In the Give-N task, children were given 15 counters and asked to give a specific number of counters (e.g. "Give me five counters, please"). The requested numbers covered 1 to 8 and 0. Zero was always administered as a number word not as a linguistic term describing zero. All numbers were requested in three trials each as suggested by Sarnecka and Lee (2009). Numbers were randomized in the three trials to avoid position effects. In the three trials, counters were changed (e.g. stones, candy, toys). The internal consistency of the Give-N tasks was good both for the natural numbers only (Cronbach's α = .856) and including zero (Cronbach's α = .854).

Ordinal Concepts of Zero (OCZ): The ordinal number concept for zero was assessed with eight tasks (Fritz et al., 2018). Two of the tasks each were assigned to one of the four aspects of the ordinal concepts: ordering (e.g. "Which number comes before one?"), comparing (e.g. "Which number is smaller – five or zero?"), addition (e.g. "You have three and get zero more – how many do you have now?"), and subtraction (e.g. "You have seven and give away zero – how many do you have now?"). The instructions of these tasks were given verbally with the usage of the term zero in all cases. Internal consistency for the eight items was good (Cronbach's α = .77).

The tasks were derived from more detailed tests for early arithmetic concepts of natural numbers (Ricken et al., 2013). In the original version, the items refer to natural numbers (e.g. "Which number comes after three?"). The Rasch model underlying the original test confirmed that the original items constitute a unidimensional scale, that is describe one arithmetic concept (Fritz et al., 2018; Ricken et al., 2013).

Results

Analysis strategy

According to Sarnecka and Lee (2009), children's counting skills can be classified by the highest number that they can reliably produce in the Give-N task. A number is produced reliably when (a) the child produces the correct number at least in two out of three trials, (b) the lower numbers are also produced correctly at the same benchmark, and (c) if the number is not produced when asked for higher numbers. Based on the children's answers in the Give-N task, they were categorized into a Knower-level that corresponds to the highest number they could reliably produce. Analogously, children's knowledge of zero was determined ("Zero-knowers"). Children, whose knower-level was bigger than three were also categorized as being cardinal-principle-knowers (CP-knowers), whereas children with a lower knower-level were categorized as subset-knowers.

Based on theoretical and empirical findings, children's understanding of zero and natural numbers can be assumed to develop in the form of overlapping waves (Clements & Sarama, 2014; Siegler & Alibali, 2005). The model of overlapping waves assumes that numerical competence does develop in phases that can be described by specific strategies or response patterns. However, these phases do not separate into distinct steps, but overlap. Thus, a child at a specific phase (e.g. two-knower) is characterized by giving exact two items when asked to, but random items when asked for a number bigger than two. Nonetheless, this specific child occasionally might be able to give three or four items when asked to, or fail when asked for one or two items.

As the current study aims at investigating children's understanding of zero in relation to their understanding of natural numbers, the overlapping waves model appears appropriate for data analysis. Previous studies have successfully employed one-dimensional Rasch-models to measure competence development within the overlapping waves framework in different contexts (Clements et al., 2008; Fritz et al., 2018; Herzog et al., 2019; Schulz et al., 2020). Here, two one-dimensional polytomous Rasch-models – one including the Give-N tasks for numbers 0-8 and one including the Give-N tasks and the ordinal concept of zero – will be used to address both research questions. All analyses were conducted using R (R Core Team, 2018) and the package TAM (Robitzsch et al., 2021).

The Rasch-model is a probabilistic model that measures both ability of the participants and difficulty of the items on one scale. Based on the item difficulty measures, a hierarchy in their development can be investigated. In this study, a polytomous Rasch-model was employed that gathered the responses in the Give-N tasks for each number including zero, as well as the ordinal concept of zero. In a polytomous model, several answer categories are combined. In this case, the categories were characterized by the number of correctly answered trials. The overall difficulty of the tasks is expressed by the beta-value. The discrimination between children with few and many correct answers in the trials is expressed by the alpha-value. The degree of fit of the data to the model is (besides others) expressed by the MNSQ-infit values. Infit values less than 1 indicate a redundancy in the items, infit values bigger than 1 indicate that the items do not measure the same construct. Wright and Linacre (1994) defined a range of .7 to 1.3 as sufficient.

Children's understanding of zero

In this study, 39 children were categorized as subset-knowers and 68 children as CP-knowers. A total of 38 children were zero-knowers, of which the vast majority of 35 were also CP-knowers. The relation of counting skills and knowledge of zero gets even more visible when considering the percentages: Only 8.3% of the subset-knowers were zero-knowers, but 51.5% of the CP-knowers. A Chi-square test confirmed the statistical significance of the difference in distribution ($\chi^2(1)=20.742$, p < .001). Focusing on the CP-knowers, seven-knowers and eight-knowers had the highest percentages of zero-knowers. More than 73.7% of the zero-knowers were at least seven-knowers. However, one third of the seven- and eight-knowers in this study had not yet developed an understanding of zero.

Mean age of the children increased with increasing knower-level. However, the age increase across knower-levels is not constant. A one-factorial ANOVA confirmed general age differences between the knower-levels ($F(8, 98) = 3.885$, $p < .001$, $\eta^2 = .241$), but

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Bonferroni-corrected post-hoc tests only confirmed age differences between two-knowers and seven-knowers as well as eight-knowers. Zero-knowers were eight months older on the average than non-zero-knowers ($F(1, 105) = 34.698$, $p < .001$, $\eta^2 = .248$). Mean age of the zero-knowers was 62.9 months (*SD* = 7.57 months, range: 48.93-78.30) and thus even higher than the mean age of the eight-knowers. Children's classification into knower-levels and the corresponding mean ages are summarized in Table 1.

Table 1

Knower-levels and mean ages

Knower-Level	Children	Zero-knowers	Age
	n	*n (%)*	*M (SD)*
Subset	39	3 (8.3%)	53.49 (6.65)
0	3	0 (0%)	50.99 (2.25)
1	5	0 (0%)	55.47 (9.57)
2	19	1 (5.3%)	51.32 (4.87)
3	12	2 (20%)	56.71 (7.54)
CP	68	35 (51.5%)	59.97 (7.65)
4	7	2 (40%)	55.85 (3.92)
5	2	0 (0%)	58.79 (5.54)
6	13	5 (38.5%)	57.64 (6.07)
7	14	9 (64.3%)	60.08 (9.90)
8	32	19 (59.4%)	61.85 (7.64)

**Note: Subset=Subset-knowers; CP=Cardinal principle-knowers; n=subsample size; M = mean; SD = standard deviation.*

The Relation of Zero to Natural Numbers up to Eight

To address the first research question, a first Graded Partial Credit Model (GPCM) was employed based on the responses of the Give-N task for numbers one to eight and zero. Each number was asked in three trials, which leads to four response categories ranging from 0 to 3 correctly answered trials. The MNSQ-infit values of the first GPCM ranged between .73 and 1.29 for all items and categories, which is considered acceptable (Wright & Linacre, 1994). The EAP reliability of the model was .832 and therefore good.

Item parameters of model 1 are summarized in table 2. For numbers 1 to 8, beta values increased successively, indicating that bigger numbers were more difficult in the Give-N task. Differences between the numbers were bigger for numbers 1 to 4 (minimum = .235 logits, range=10.817 logits) and smaller for numbers 5 to 8 (minimum = .012 logits, range=.350 logits). All items discriminated relatively strongly between children with high and low counting ability. This finding suggests that the natural numbers up to eight form consistent competencies.

Compared to the natural numbers 1 – 8, zero was substantially more difficult. Moreover, zero differentiated less between children with high and low counting ability. Thus, zero seems to be not as consistent as the natural numbers up to eight.

Table 2

Parameters of the GPCM models.

	Model 1		Model 2	
Item	Alpha	Beta	Alpha	Beta
One	2.218	-11.493	2.109	-2.548
Two	1.960	-1.963	2.120	-1.842
Three	2.922	-.911	omitted	omitted
Four	2.038	-.676	2.668	-.406
Five	2.823	-.200	3.018	-.053
Six	1.745	-.185	1.936	.063
Seven	2.993	.054	2.363	.232
Eight	2.151	.150	2.451	.348
Zero	.579	.731	1.047	.617
OCZ_ord	-	-	.678	.698
OCZ_com	-	-	.688	-.023
OCZ_add	-	-	.599	1.163
OCZ_sub		-	.790	1.249

**Note: OCZ_ord = ordinal concept of zero, subskill ordering; OCZ_com = ordinal concept of zero, subskill comparing; OCZ_add=ordinal concept of zero, subskill addition; OCZ_sub = ordinal concept of zero, subskill subtraction*

The Relation of the Ordinal Concept of Zero to the Meaning of Zero

To address the second research question regarding the relation of the ordinal concept of zero and an understanding of the meaning of zero, the subskills ordering, comparing, addition, and subtraction were added to a second GPCM. To avoid distortions in the GPCM caused by varying category numbers, categories of the Give-N task were adapted to three categories as provided by the OCZ tasks. For this reason, the categories for 0 and 1 correctly answered trials were collapsed to one category.

With one exception, the MNSQ-infit values of the initial second GPCM ranged between .79 and 1.25 for all items and categories. Only item "Three" showed insufficient infit values (.58) and was therefore omitted. The remaining items in the final second GPCM had good MNSQ-infit values ranging from .81 to 1.22 for all categories. The EAP reliability of the final model was .827.

Item parameters of the final second model are summarized in table 2, too. As in model 1, numbers 1 to 8 increased in difficulty. While numbers 1 to 4 were more distinct in difficulty, numbers 5 to 8 had closer difficulty measures. Again, the natural numbers strongly discriminated between children with high and low ability as expressed in the values of alpha. In line with the results from the first GPCM, zero was more difficult and discriminated less regarding children's ability than the natural numbers.

Regarding the OCZ, ordering was slightly more difficult than the understanding of zero. Addition and subtraction as subskills of the OCZ were substantially more difficult than the understanding of the meaning of zero. Against expectancies, comparing numbers was relatively easy and not substantially more difficult than the counting competency of the CP. Especially was the comparing facet of the OCZ less difficult than the understanding of the meaning of zero. All items measuring the OCZ had very low alpha-values, indicating that the development is less consistent than that of the natural numbers or zero.

Discussion

The aim of this study was to investigate how the understanding of zero in counting and ordinality is related to the understanding of the natural numbers. The results of the current study show that an understanding of zero as a counting reference to an empty set is harder to understand than the natural numbers up to eight. Both the parameters and the age differences between zero- and non-zero-knowers support this notion. This finding is in line with previous studies that found that Cardinal Principle-knowers (CP) are more proficient in the understanding of zero than children, who only know the numbers one, two or three (subset-knowers) (Krajsci et al., 2021; Pixner et al., 2018). In contrast to Pixner et al. (2018), we found substantial age differences between zero-knowers and non-zero knowers. The findings go even beyond: Obviously, the CP-knowledge is not sufficient, as illustrated by the substantial difference in difficulty between zero and four in the Rasch models and the fundamental skewness in the distribution of zero-knowers across subset-knowers and CP-knowers. More experiences with even more numbers are needed to consider zero as a number. This raises the question, to which extent the cardinal principle is the adequate framework for zero. Or, in other words, is the cardinal principle the only relevant knowledge children need to understand zero?

Doubts regarding the relevance of the cardinal principle for the understanding of zero may be grounded in the different mechanisms underlying the learning processes of counting in natural numbers and zero: Whereas natural numbers have a referent (the number word) and a reference (the corresponding set) that can be mapped: "Four" refers to a set of four items. However, in the case of zero, there is a referent (the word "zero"), but no visible reference, since there is no item. But how can an empty set be represented? Thus, there might be a qualitatively different process responsible for the development of understanding zero. Empirical evidence in support of this notion can be found in the differences in the discrimination between more or less able children of understanding the natural numbers and zero in the Rasch model, which might indicate qualitatively different learning processes. We therefore assume that the main developmental driver for understanding zero results on the one hand from the concept of ordinal representation of numbers and on the other hand from the matching linguistic concepts. Since the semantic concept of zero is double abstract, meaning no visible reference point, and no anchor in the core systems, it must be constructed via multiple avenues of access. In other words, zero does not seem to be a "natural" number, if "natural" is determined the way, that the relation of the number word, its magnitude and its visible reference can be mapped onto each other.

Regarding the relation of the OCZ and the understanding of zero, results were inconsistent. The operations addition and subtraction were substantially more difficult than the understanding of zero in this study. These findings suggest that the operation aspect of the ordinal number concept is based on counting knowledge both for natural numbers and zero. A closer look at the processes involved reveals that operations require an understanding of numbers in the context of counting. Addition by counting does not work differently for natural numbers and zero.

In contrast to the operation aspect of the ordinal number concept, comparison was less difficult, and even easier than the understanding of zero. This means that children were more likely to locate zero within the number word sequence than to give zero items. This finding contradicts the findings for natural numbers that number comparison is based on counting proficiency (Le Corre, 2014). In this sense, zero seems to be different from the natural numbers. Against the background of a potentially qualitatively different developmental path to understanding zero, the ordinal understanding of zero as a predecessor of the number 1 might be a driver of development.

Based on the theory of Carey`s bootstrapping process there is a need to actively construct zero as the starting point of the number word sequence. Since the number word sequence is not learned starting with zero but always goes from one up to ten, the concept of zero does not start with a placeholder function like all other natural numbers. Thus, perhaps semantic – numerical information is first constructed via bootstrapping to all placeholding number words up to 4. After that, counting processes take over for numbers greater than 4. As these surface concepts develop, ordinal aspects form. Here, ordering and comparing come first.

Now the problem is to find a suitable place in the number line for the number zero. Semantic terms that express nothing are helpful here because they indicate that zero is even smaller than one. Comparisons of all kinds of linguistic expressions for empty sets with one or more objects lead children to place zero still before one. Perhaps children first need to understand the successor and predecessor functions to develop an understanding that zero is the predecessor of one.

At this point, children can apply the predecessor function to the counting routines: If zero comes before one, it has exactly one item less, resulting in an empty set. Such a developmental path would mimic the assumptions of the successor function as a developmental driver for the cardinal principle in reverse (Carey, 2009). The proposed development of the understanding of zero is in line with the results of the Rasch model, in which the ordering aspect of zero was only slightly more difficult than the counting knowledge of four, identified with CP knowledge. This could lead to the interpretation that the representation of zero may be tied to its ordinal position rather than to the very abstract cardinal representation of an empty set.

Beyond the research questions, the increasing and pronounced difficulties of numbers one through four, as located on the difficulty continuum shown by the application of the Rasch model provides additional evidence for the assumption that the natural numbers up to four are successively developed (Negen & Sarnecka, 2012). However, with respect to numbers five through eight, the results can be interpreted in two ways: First, the smaller difficulty gaps between numbers five through eight could indicate that counting knowledge of these numbers is associated with increasing conceptual knowledge. The slightly increasing difficulties between five and eight are due to the longer counting processes, which are more prone to random errors. On the other hand, the increasing difficulties as reflected by the Rasch model show that these numbers, like numbers one through four, are learned hierarchically and successively. However, the development of numbers five through eight could be accelerated by more routine, which would explain the decreasing differences between the difficulties of the numbers. Accelerated development with increasing numbers could be the reason why previous studies have not found significant differences in counting skills between these numbers: Children who have understood the meaning of the number four are likely to know larger numbers, as understanding of the numbers five through eight can be very rapid.

The first interpretation supports the construct of the CP-knowledge. The second interpretation suggests that numbers bigger than four are not conceptually embedded in the CP-knowledge, but that these numbers are also learned successively. Further research – especially longitudinal studies –might inform the proposed interpretations. However, the first interpretation can be better brought in line with the literature at the moment.

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Statistics Anxiety in Flanders: Exploring Its Level, Antecedents, and Performance Impact Across Professional and Academic Bachelor Programs in Psychology

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Abstract

The current study focusses on the level of statistics anxiety and the motivation to learn statistics in Flanders (Belgium) and determined to what degree these factors and their interaction relates to statistical performance. For this purpose, the Statistics Anxiety Scale and the Statistics Motivation Scale were translated, validated, and administered in professional and academic bachelor students in psychology. The level of SA in Flanders is comparable to other countries, with professional bachelor students being more anxious to make interpretations compared to academic bachelor students, who in turn are more anxious to ask for help. Academic students are more motivated to learn statistics compared the professional bachelor students, mostly in terms of intrinsic motivation. The overall motivation to learn statistics is lower at the end of the semester compared to the beginning of the semester. This is unfortunate, because we observed that high levels of motivation can alleviate the negative impact of statistics anxiety on statistical performance, especially when controlling for general learning abilities.

Keywords:

Statistics Anxiety, Motivation, Flanders, Statistical Performance

Introduction

Statistics courses are a pivotal component of many college and/or university programs. Besides the direct application of statistical knowledge for research purposes, insight into statistics is more generally considered an important steppingstone in the development of critical thinking, and decision and problem-solving skills (Kesici et al., 2011). For a variety of reasons, students typically find their statistics course to be the most anxiety-inducing course in their study program (Caine et al., 1978; Zeidner, 1991). Critically, statistics anxiety (SA) has been claimed to negatively impact the statistics learning curve and ultimately, statistics performance (Macher et al., 2012, 2013, 2015; Onwuegbuzie & Wilson, 2003; Papousek et al., 2012; Zare et al., 2011). To better understand and manage SA, the phenomenon needs to be explored in terms of its prevalence, antecedents, and impact on student performance. This would be useful information for teachers, lecturers and educational policy makers who are involved in facilitating and increasing the level of "data-literacy" at schools and (more generally) in society. In the current study, we explore these elements in the context of higher education in Flanders.

Prevalence and Impact

Defined as an anxiety that occurs when encountering statistics in any form and at any level (Onwuegbuzie et al., 1997), the study of SA is critically dependent on valid and reliable assessment tools. Initially, a popular framework in the context of SA - the six-factor model of Cruise et al. (1985)- considered SA as a multidimensional construct composed of "interpretation anxiety", "test and class anxiety", "fear of asking for help", "computational self-concept", "worth of statistics", and "fear of statistics teachers". Reflecting these factors, a 51-item Statistics Anxiety Rating Scale was proposed to measure SA (Cruise et al., 1985), and was the dominant measure used in the literature for a long time (Cui et al., 2019). Later studies, however, suggested that only the first three subscales of this model (interpretation anxiety, test and class anxiety, and fear of asking for help) are direct indices of SA, whereas the latter three subscales (worth of statistics, computation self-concept, and fear of statistics teachers) assess attitudes towards statistics rather than SA (Chew & Dillon, 2014b; Papousek et al., 2012). To remedy this, a shorter instrument was developed by Vigil-Colet and colleagues (2008) coined the Statistical Anxiety Scale (SAS) – thus taking only three of the original six factors into account. Where the SAS was initially developed and validated in Spanish, the instrument retained its good psychometric properties after being translated to be used in other countries (e.g., Italy, Australia, Singapore, Bangladesh, and the USA). To our knowledge, a Dutch (the official language in Flanders) translation of this questionnaire has not been validated so far (making it one of our research aims; see Research Aim 1 below).

Using the above instruments, SA has been shown to be broadly presenting itself across various countries and their respective educational systems. In their study, Zeidner (1991) found that as many as 70% of Israeli students experienced SA. Similarly, Onwuegbuzie and Wilson (2003) estimated that about 80% of graduate students in Georgia (USA) experience uncomfortable levels of SA. Furthermore, students in the social sciences (e.g., psychology) are especially prone to report high levels of SA (Zeidner, 1991). This may be because these students typically had relatively few hours of mathematics in their high school program, and/or had negative prior experiences with mathematics in high school – both of which are known to be potential antecedents for the development of SA (Onwuegbuzie & Wilson, 2003). Indeed, SA was initially thought to largely overlap with the math anxiety (Mitton, 1987). Yet, despite SA being related to math anxiety (with correlations generally in the range of r = .40-.70), there is a consensus that these constructs refer to distinct phenomena (Baloğlu, 2002; Benson, 1989; Paechter et al., 2017). This warrants a dedicated study of SA.

Efforts to map out the level of SA through valid assessment tools, is motivated by the impact SA has on the way how students are engaged in studying statistics. For example, SA has been associated with procrastination of learning (Onwuegbuzie, 2004), spending less time on studying, and the use of less efficient learning strategies (Macher et al., 2012, 2013). As a result, SA is often considered as a major negative influence on the performance in statistics courses (Onwuegbuzie, 2004). Despite this, studies that directly investigated the link between SA and statistical performance are less univocal. Where several studies reported a small but significant negative correlation between SA and statistical performance (typically ranging between r = -.20 and r = -.30; for an overview see Macher et al., 2015) other studies demonstrated insignificant or even positive correlations (Lester, 2016; Paechter et al., 2017).

Macher and colleagues (2015) explain these contradicting results on the relationship between SA and performance by making a distinction between the direct and indirect effects of SA on performance. A direct link between SA and academic performance pertains to the moment of examination. Anxiety leads to an increase in task-irrelevant thoughts (such as worry or rumination), which reduce the cognitive resources that are necessary to successfully complete the statistical problems (Eysenck et al., 2007). While such direct effects are typically negative, indirect effects can be both positive and negative. For example, SA can have an indirect negative effect on performance via difficulties in time-management and procrastination (Onwuegbuzie, 2004; Rodarte-Luna & Sherry, 2008), but SA can also be positively related to performance via increased effort and motivation when the level of anxiety is manageable (Dunn, 2014; Macher et al., 2015). Importantly however, whereas compelling evidence exists for indirect negative effects, support for the idea that effect of SA on performance can be moderated by the motivation to learn statistics is less established (see Research Aim 4 below).

Besides the complex link between these and other mediating/ moderating factors, it is also possible that the lack of consistent results is due to the lack of proper control variables. Indeed, someone's grades on the statistics exam will be caused by many other variables as well. Among other factors, general intelligence, general curiosity, psychological and physical wellbeing during the (preparation of the) exam can also have an influence. Interestingly, it can be predicted that (some of) these factors are also correlated with someone's mathematical background or motivation to learn statistics. It would therefore be interesting to reinvestigate the link between SA and statistical performance when controlling for this "learning efficiency" factors.

Antecedents of Statistics Anxiety

Several studies have tried to shed a light on the origins of SA. Overall, these antecedents can be categorized into three major factors: situational, dispositional, and cognitive factors (Cui et al., 2019; see Chew & Dillon, 2014 for another classification). Situational factors are present in the external environment or in situations that are related to SA. These include, but are not limited to properties of the curriculum format and teaching styles like e.g. the pace of statistics instructions (Bell, 2005 as cited in Chew & Dillon, 2014b), the class atmosphere (Lesser & Reyes III, 2015), the absence of real-life examples (Neumann et al., 2013), instructor immediacy (Tonsing, 2018), the verbal and/ or non-verbal expressions of the lecturer (Williams, 2010), or the organization (online vs on campus) of the courses (DeVaney, 2010). Given that these factors are to a large extent determined by specific educational systems, it is of relevance to map out the levels of SA across different countries and their different educational systems – with the focus of the current study (Flanders) not yet having been explored in this context (see section below, see Research Aim 2 below).

Dispositional factors are factors that the student brings into the setting and are related to individual differences in, e.g., the attitude towards statistics, the motivation to learn it, prior mathematical experience, or procrastination behavior. With respect to attitude and motivation, it has been found that a negative attitude towards statistics is associated with higher levels of SA (Schau, 2003), and that positive attitudes towards statistics can diminish the negative effects that SA has on statistical performance (Najmi et al., 2018). Relatedly, intrinsic interest in the subject is associated to lower levels of SA. This is probably because interested students show more cognitive engagement when studying and more frequently use efficient learning strategies (Macher et al., 2012). Personal experience with mathematics also plays a role in the development of SA. An insufficient mathematical background, bad experience with math, and math anxiety are all related to SA (Abd Hamid & Sulaiman, 2014; McGrath, 2014). This could be because SA often gives rise to procrastination and vice versa. Higher levels of SA are found in people who procrastinate (due to a general fear of failure or trait anxiety), and higher levels of SA often result in procrastination (Chew & Dillon, 2014b). Additionally, procrastination is often associated with less efficient learning strategies (Vahedi et al., 2012). As such, the relation between procrastination and SA is often characterized as a downward spiral, where procrastination and inefficient learning often lead to bad experiences with statistics (which could trigger SA), which in turn gives rise to procrastination behavior. Another important dispositional factor is the cultural background of the students, as the degree of SA (and probably the link with statistics performance) differs between countries and between subgroups within countries. For example, as a group, Chinese students show lower levels of SA compared to students of the USA and UK (Liu et al., 2011), while international students (in the USA) seem to suffer from higher level of SA compared to their domestic counterparts (Bell, 2008). Although the precise mechanisms underlying these differences are still unclear (Cui et al., 2019), they are typically attributed to differences in the educational system (e.g., whether or not it is common to ask questions during class), the mathematical background of the students (incl. the habit to use calculators), or whether the classes were given in the first language of the students or not (Bell, 2008).

Apart from the situational and dispositional factors, there are also cognitive factors. These factors refer to the cognitive resources (like working memory and executive functions) that are recruited when solving statistical problems. Several attempts have been made to identify the cognitive factors that help us to explain individual differences in SA and the relation with the performance on statistical tasks. In this context, SA seems to be related to basic numerical abilities (Paechter et al., 2017), metacognitive abilities, effective inhibition of task irrelevant information, and verbal reasoning abilities (for an overview see Cui et al., 2019). This field, however, is relatively new and further studies will be needed to identify and to further describe the role of these (and other) cognitive functions as antecedents of SA.

Statistics and Statistics Anxiety in Flanders

In Flanders (Belgium), statistics is also an important course in the Psychology program with the mathematical/ statistical background often being an important element for a student to decide to enroll in the program or not. Compared to (some) other countries, the way how the Psychology programs are organized in Flanders is rather unique. In Flanders, the bachelor's in Psychology has programs at the academic level (at universities) as well as at the professional level (at university colleges). Whereas the academic bachelor prepares students with the appropriate (scientific) knowledge and skills to continue in the master's program, the professional bachelor's program focuses on professional practice providing students with competencies (knowledge, skills, and attitudes) to directly start a profession (e.g., in psychodiagnostics, counselling and/ or coaching, selection and recruitment, etc.).

In their information brochures for novel psychology students, both universities and university colleges explicitly refer to the importance of statistics in their curriculum and the recommended level of mathematical background. Although no explicit mathematical level is required to enroll, higher levels of mathematical background are recommended for the academic bachelor's in Psychology compared to the professional one. For example on the website for future students, the Catholic University of Leuven recommends a (high school) background of 9 hours of mathematics and science a week (https://onderwijsaanbod.kuleuven.be), and Ghent University encourages students who had less than 5 hours/ week of mathematics in high school to test their level of mathematics beforehand, and to consider following an online math course in case their mathematical background turns out to be insufficient (https://www.ugent.be/pp/nl/toekomstige-student/). University colleges on the other hand, recommend lower levels of mathematical background. Thomas More University college recommends a background of minimum of 3 hours of math a week (https://www.thomasmore.be/opleidingen/professionele-bachelor/toegepaste-psychologie/toegepaste-psychologie-een-goede-start), while no recommendation is given by the AP university college (https://www.ap.be/opleiding/toegepaste-psychologie). Therefore, students who received less hours of math per week in high school, as well as students with negative experiences with or attitudes towards mathematics are often more inclined (and sometimes even encouraged) to go to college rather than university. Furthermore, besides in the recommended mathematical background, university colleges and universities also differ in the way how their education is organized. Typically, at the university college the lecturer-student distance is smaller, and courses are organized in smaller groups, with more attention for student coaching (www.onderwijskiezer.be). In universities, teaching occurs in large groups, not seldomly comprising several hundreds of students. To our knowledge in Flanders, SA and the motivation to learn statistics in psychology students have not yet been investigated, especially not when considering differences between professional and academic bachelor students (see Research Aim 3).

Mapping out the level of SA and the motivation to learn statistics in Flanders offers several opportunities. Besides giving an impression about the numbers of students suffering from SA and about their motivation (which could be interesting for lecturers in statistics and future students), the direct comparison between professional and academic bachelor students would be an ideal opportunity to further investigate the influence of dispositional factors on the prevalence of SA in students with a shared study interest (psychology), a shared (first) language and a common educational (high school) culture.

Additionally, the link and interactions between SA, the motivation to learn statistics, mathematical background, and the scores on the exam of statistics are also worth exploring. Studies in Flanders so far mainly focused (in university students) on the link between mathematical background and academic success and found that students with stronger math backgrounds have higher chances to successfully pass their exams in statistics (and have higher chances for academic success in general; Fonteyne et al., 2015). From the studies summarized above, we know that SA and the motivation to learn statistics could be mediating/ moderating factors as well, but more empirical efforts are needed to come to a fuller understanding of how these factors and the interaction between them impact statistical performance. Given the less stringent recommendations with respect to prior mathematical experience for the professional bachelor programs, more heterogeneity can be expected in terms of mathematical background, SA and the motivation to learn statistics (which is ideal when studying correlations and interactions between variables). Afterall, it can be expected that the professional bachelor's in Psychology contains both students with a primary interest in this option (with or without a low mathematical background, SA and a low(er) motivation to learn statistics) but also students who chose for this option, because they question their statistical competences to start the academic bachelor.

Research Goals of the Current Study

The current study was designed to investigate SA and the motivation to study statistics in Flanders. For this purpose, the Statistics Anxiety Scale (Vigil-Colet et al., 2008) and the adapted Academic Motivation scale (Vallerand et al., 1992) were translated into Dutch and administered to first year students enrolled in the professional or academic bachelor's in Psychology program who were for the first time taking the course of "Statistics 1" (the introductory course of statistics).

The aim of the current study was four-fold. The first research goal is to see whether the Dutch translation of the Statistics Anxiety Scale (Vigil-Colet et al., 2008) and the Statistics Motivation Scale (adapted from, Vallerand et al., 1992, see below) were sufficiently reliable and valid. Besides internal consistency of the outcomes, we also used factor analyses to see whether we could replicate the factorial structure which is typically found in the original version and in other translations. Additionally, we performed several correlations to see whether we could replicate the previously reported relations between SA and motivation (Schau, 2003), grades on the exam (Macher et al., 2015), hours of mathematical background (Abd Hamid & Sulaiman, 2014; McGrath, 2014) and math anxiety, to determine whether we could confirm the claim that statistics anxiety is related but different from math anxiety (Baloğlu, 2002; Benson, 1989; Paechter et al., 2017). The second and third goal is to get an idea about the average level (and the distribution) of statistics anxiety and about the motivation to learn statistics in Flemish psychology students in general and (the third goal) to investigate whether there are differences between the professional and academic bachelor students. Finally, the fourth aim is to investigate whether and how SA, the motivation to learn statistics and their interactions are related to the scores on the statistics exam when controlling for general learning abilities and whether they explain additional variance on top of someone's mathematical background. Given the different recommendation for enrollment in both programs, we predict that students in the professional bachelor's program show higher levels of SA and a lower motivation to learn statistics than students in the academic bachelor's program. Because the learning environment of university colleges could act as a protecting factor to develop SA (smaller lecturer-student distances, more attention for student coaching), we predict that the difference in SA between both bachelors will be more pronounced in the first weeks of the semester before these protecting factors have had the chance to become effective. For this purpose, two data-collection waves were organized, one in the beginning of the semester (September), and one at the end (December). Fourth and finally, we predict that besides mathematical background, SA, and the motivation to learn statistics and/ or their interaction also explain a significant proportion of variance in the scores on the statistics exam.

Methods

Participants

The participant pool consisted of 438 first year bachelor students in Psychology who were all for the first time enrolled to Statistics 1 as undergraduate. All students were between 18 and 20 years old. The sample consisted of 272 (62.1%) professional bachelor students taking classes at the Thomas More University of Applied Sciences and 166 (37.9%) academic bachelor students at Ghent University. 298 students gave permission to collect the scores of their statistics exam and the exam of general psychology. The exam scores of 66 participants where not considered, as they also participated (after completion of the questionnaires) in another study which included a psycho-education session about SA and exercises to stimulate a growth mindset. Furthermore, for General Psychology, only the grades from the Thomas More students (who gave their permission, n = 180) were accessible. Two waves of data-collection were organized, which included different subjects (in other words, a between subject design was used). The first wave took place from the end of September till mid-October 2020 (n = 187). The second wave was organized from the end of November till mid-December 2020 (n = 251). A detailed description of demographic information (age, gender, and number of participants from the different institutions, at the different test moments) can be found in Table 1. All students provided their informed consent beforehand and participated either without any compensation or for course credits. The study was approved by the ethical committee of the Faculty of Psychology of Ghent University.

Materials

Three self-report questionnaires were presented: (1) the Statistical Anxiety Scale (2) the Abbreviated Math Anxiety Scale and (3) the Academic Motivation Scale. Using the back translation method (Beaton et al., 2000), the questionnaires were translated into Dutch, since no versions of the questionnaires were available in this language. The translated questionnaires were reviewed by a content expert, a language expert, and two researchers and were also assessed by them for face validity. The translated versions of the questionnaires can be found in Appendices A - C

Statistical Anxiety Scale (SAS)

The Statistical Anxiety Scale was used to measure statistics anxiety (Vigil-Colet et al., 2008). The SAS consists of three subscales, containing eight items each, assessing different aspect of statistical anxiety. The "Examination Anxiety" subscale assesses the anxiety experienced during a statistical examination. The "Anxiety for Asking Questions" subscale assesses the anxiety students may feel when asking statistics related questions to the course teacher, another student, or a private teacher. Finally, the "Interpretation Anxiety" subscale aims to measure the anxiety experienced when students interpret statistical data and understand the formulation used in statistics. Participants were instructed to indicate on 5-point Likert scale (1 = low anxiety to 5 = a lot of anxiety) how anxious they would feel during the described situations involving statistics (e.g., 'Studying for an examination in a statistics course'). Besides these subscales, a general SA index was determined by calculating the average of all 24 item scores. The SAS was constructed in the context of the introductory statistics course for undergraduate psychology students. Previous research indicated good reliability and validity in other languages as well (Cantinotti et al., 2017; Chiesi et al., 2011).

Abbreviated Math Anxiety Scale (AMAS)

Math anxiety was assessed using the Abbreviated Math Anxiety Scale (Hopko et al., 2003). Participants were asked to indicate on a 5-point Likert scale (1 = low anxiety, 5 = high anxiety) how anxious they would feel in each of the nine described math situations. The average score on these nine items was calculated as an index, with higher values corresponding to higher anxiety. The AMAS is characterized by adequate reliability and validity in the original English version as well as in several other languages (Caviola et al., 2017; Cipora et al., 2018). The original English version had an internal consistency of $\alpha = .90$ and a test-retest reliability of rtt = .85 (e.g., Hopko et al., 2003).

Statistics Motivation Scale (SMS).

Finally, to measure the motivation to learn statistics, we adapted the Academic Motivation Scale. This scale is based on the tenets of self-determination theory and measures "extrinsic motivation", "intrinsic motivation" and "amotivation toward education" (Vallerand et al., 1992). To our knowledge, a modified version adapted to the context of statistics has not yet been developed. However, recently the scale has been adapted to the context of mathematics (e.g., Staribratov & Babakova, 2019). To measure academic motivation towards statistics, we started from this math version and replaced the word 'math' with the word 'statistics' in each item of this scale. This modified AMS (hereafter called the Statistics Motivation Scale; SMS) consisted of a total of 15 items (seven, five and three items for the extrinsic motivation, intrinsic motivation and amotivation subscales respectively). Participants were then asked to rate to what extent the statements describe why they study statistics. Each item was measured on a scale from 1 (not at all) to 7 (completely). An example of such a statement is: 'I'm motivated to study statistics.... because statistics lets me discover many new and interesting things.' Besides the subscales, a general motivation index was determined by calculating the average of all 15 item scores. This scale has been found to have a good reliability and validity (Vallerand et al., 1993).

Procedure

All questionnaires were distributed using Qualtrics or Lime survey and were provided to the participants via an e-mail link. A call for participation was announced during the courses of General Psychology (Thomas More) and Statistics 1 (Ghent University) for those subjects who participated freely, or via the course credit website, for those who participated to obtain course credits. The following demographic variables were collected: (1) age, (2) gender, (3) number of hours of math per week in their last year of high school, (4) study program in high school and (5) current study program. Subsequently, the three questionnaires were presented in the following order: SAS, AMAS then SMS. Before beginning the SAS and SMS, participants were instructed to think about their experiences with the "Statistics 1" course of their current study program for answering the questionnaire. Before beginning the AMAS, participants were asked to think about their experiences with a math course in high school when rating the items. The time to complete the questionnaires was approximately 11 minutes. Two waves of data-collection were organized (including different subjects). Importantly, due to restrictions related to the covid-pandemic, all classes (of both the professional and academic bachelor) were organized online from November 1st, 2020. As such, all students were following online classes during the second wave of data-collection.

Results

Descriptive Statistics

A full overview of the descriptive statistics of all variables and demographical information can be found in Table 1. Summarized: As typical for psychology students, most of our sample consisted of females (85%). As expected, on average, the professional bachelor students had less hours of mathematics in high school compared to the academic bachelor students [3.23 hours per week, $SD = 0.93$ vs. 3.99 hours per week, $SD = 1.41$; $t(433) = -6.85$, $p < .001$]. The group of professional bachelor students was also a bit older [18.49 years, $SD = 0.67$ vs. 18.23 years, $SD = 0.52$; $t(433) = -4.30$, $p < .001$]. Importantly, the groups of the first and second data-collection wave did not differ from each other in terms of age and hours of mathematical background [both t's(433) > -1.53, p's > .06].

Q1: What is the Reliability and (construct) Validity of the Dutch Translation of the Statistics Anxiety Scale, and the Statistics Motivation Scale?

Statistics anxiety scale (SAS)

Because the SAS has never been translated to Dutch and used in a Flemish population, it was decided to evaluate the construct validity in three steps. First, an exploratory factor analysis (EFA) was performed to verify the factorial structure of the test. Next the factors found with the EFA were further checked using confirmatory factor analysis (CFA; for a similar approach, see Durak & Karagöz, 2021). All FAs were conducted using JASP (Love et al., 2019). For the EFA, maximum likelihood estimation was conducted on the 24 items of the SAS with varimax rotation. The Kaiser-Meyer-Olkin (KMO) measure was used to verify the sampling adequacy for the analysis. The overall

Table 1
Descriptive statistics

		Professional bachelor						Academic bachelor					
		begin semester			end semester			begin semester			end semester		
Gender	female	96			129			57			92		
	male	24			22			8			7		
	other	1			0			1			1		
		average	*SD*	*n*	*average*	*SD*	*n*	*average*	*SD*	*n*	*average*	*SD*	*n*
Age	(years)	18.33	0.57	121	18.62	0.72	151	18.32	0.56	66	18.17	0.47	100
Mathematics	(hours/week)	3.21	1.02	121	3.25	0.83	151	3.79	1.23	66	4.11	1.50	100
Statistics	points/20	10.38	4.60	79	9.76	4.36	101	8.10	4.39	42	9.72	4.35	18
General Psychology	points/20	11.51	4.01	88	11.59	3.78	143	na			na		
Statistics Anxiety Scale	General	2.79	0.64	121	3.09	0.59	151	3.09	0.60	66	3.02	0.65	100
	Examination	3.93	0.75	121	4.03	0.72	151	4.09	0.66	66	3.97	0.78	100
	Interpretation	2.39	0.78	121	2.47	0.70	151	2.24	0.75	66	2.19	0.69	100
	Questions	2.61	0.97	121	2.78	0.99	151	2.94	0.98	66	2.89	1.01	100
Statistics Motivation Scale	General	3.48	0.96	121	3.27	0.97	151	3.73	1.03	66	3.53	0.96	100
	Extrinsic	3.39	1.18	121	3.13	1.18	151	3.33	1.21	66	3.15	1.25	100
	Intrinsic	2.83	1.15	121	2.64	1.19	151	3.32	1.24	66	2.98	1.16	100
	Amotivation	5.26	1.45	121	5.08	1.35	151	5.43	1.30	66	5.39	1.39	100

KMO value was 0.92 ('marvelous' according to Kaiser & Rice, 1974) and all KMO values for individual items were greater than .82 (which is well above the cutoff of .50; Kaiser & Rice, 1974). Also, the Bartlett's test was significant ($\chi^2(276) = 7138.09$, $p < .001$). Three factors had eigenvalues above 1, and in combination explained 56% of the data. Table 2A reports the factor loadings after rotation. As suggested by Field (2018), only factor loadings > .40 were considered. In this way the pattern of loadings replicated the factorial structure of the SAS that was previously reported (e.g., Durak & Karagöz, 2021). To provide further support for the observed factorial structure, an additional CFA was conducted with the same factorial structure as the one provided with the EFA, with "General Statistics Anxiety" as second order factor. The initial results did not show appropriate fit measures (the χ^2 /df, GFI, IFI, TLI, CFI, RMSEA and SRMS). An inspection of the modification indices and the misfit plot revealed a high residual covariance between Q5 and Q24 (modification index: 244.24)[1] . When the residuals of these items were allowed to correlate in the model, the model fit became statistically significant. An overview of the fit indices can be found in Table 3.

Finally, the validity of the SAS was further investigated by correlating the average SAS score with variables from which it is known that they relate to SA. Because for the SAS five correlations were calculated, the alpha level wat set to $\alpha = .01$ to correct for multiple testing and because the assumption of bivariate normality was violated [Shapiro Wilk's $p < .05$] for the correlation with the hours of mathematics in high school and the AMAS, 95% bootstrap confidence intervals were reported in Table 4. As expected, the average SAS score correlated significantly with math anxiety [$r(438) = .68$, $p < .001$], hours of mathematics in high school [$r(438) = -.18$, $p < .001$], motivation to learn statistics [$r(438) = -.23$, $p < .001$] and the scores on the statistics exam [$r(240) = -.17$, $p < .01$]. Importantly, the SAS did not correlate with the scores on the exam of general psychology [$r(231) = -.12$, $p = .06$]). A full overview of the correlations between all these variables can be found in Table 4. Finally, like previously reported (Rodarte-Luna & Sherry, 2008), we also found significant differences between males and females in their levels of general SA [$t(433) = -2.83$, $p = .005$, $d = .390$] with males *(2.83, SD* = 0.64) being less anxious compared to females (3.07, *SD* = 0.61).

The internal consistency of the average SAS score and the different subscales was determined by calculating Cronbach's Alpha. The Alpha's were .91, .89, .85 and .95 for the average score, exam anxiety, interpretation anxiety and the anxiety for asking questions subscales respectively. Taken together, it can thus be concluded that the Dutch translation of the SAS has appropriate

Table 2
Loadings exploratory factor analyses Statistical Anxiety Scale (A) and Statistics Motivation Scale (B)

(A) Statistical Anxiety Scale (In Dutch)					(B) Statistics Motivaiton Scale (In Dutch)				
	Original scale	Asking help	Examination	Interpretation		Original scale	Intrinsic	Extrinsic	Amotivation
item 1	EA		.42		item 1	EM			.50
item 2	IA			.66	item 2	IM	.75		
item 3	AAH	.85			item 3	AM		.70	
item 4	EA		.72		item 4	IM	.74		
item 5	AAH	.62			item 5	EM	.50		
item 6	IA			.67	item 6	IM	.76		
item 7	AAH	.84			item 7	AM		.83	
item 8	IA			.57	item 8	AM		.82	
item 9	EA		.65		item 9	EM			.69
item 10	IA			.63	item 10	IM	.74		
item 11	EA		.60		item 11	IM	.56		
item 12	AAH	.90			item 12	EM	.60		
item 13	EA		.80		item 13	EM			.86
item 14	EA		.61		item 14	IM	.78		
item 15	EA		.78		item 15	IM	.68		
item 16	IA			.48					
item 17	AAH	.90							
item 18	IA			.64					
item 19	IA			.57					
item 20	EA		.80						
item 21	AAH	.93							
item 22	IA			.78					
item 23	AAH	.77							
item 24	AAH	.64							
% explained variance		20%	17%	16%	% explained variance		30%	14%	12%

EA: Examination anxiety, IA: interpretation anxiety, AAH: Anxiety for asking help
EM: Extrinsic motivation, IM: intrinsic motivation, AM: Amotivation

psychometric properties to be used in a Flemish population.

Table 3
Fit indices of the confirmatory factor analyses for the SAS and the SMS

	χ^2	df	χ^2/df	GFI	IFI	TLI	CFI	SRMR
SAS	829.55	226	3.67	0.83	0.91	0.90	0.91	0.07
SMS	326.38	84	3.89	0.91	0.92	0.90	0.92	0.06
cut-off			≤5	≥0.85	≥0.90	≥0.90	≥0.90	>0.08

Statistics motivation scale (SMS)

Because of the modifications to the original instrument (Vallerand et al., 1992) and the fact that the scale has not yet been used in a Flemish sample, we again conducted both an EFA and a CFA. For the EFA, maximum likelihood estimation was conducted on the 15 items of the SMS with varimax rotation. The Kaiser-Meyer-Olkin measure was used to verify the sampling adequacy for the analysis. The overall KMO value was 0.87 ("meritorious" according to Kaiser & Rice, 1974) and all KMO values for individual items were greater than .69 (which is well above the cutoff of .50; Kaiser & Rice, 1974). The Bartlett's test was also significant [$\chi^2(105)$ = 3223.58, p < .001]. Three factors had eigenvalues above 1, and in combination explained 56% of the data. Table 2B reports the factor loadings after rotation. As suggested by Field (2018), only factor loadings > .40 were considered. In this way the pattern of loadings largely replicated the factorial structure of the original Academic Motivation Scale (Vallerand et al., 1992). With exception of items 5 and 12, which originally belonged to the extrinsic scale and now loaded on the intrinsic scale, all other items were associated with the excepted subscale. When inspecting these items, it is not so unexpected that they load on the intrinsic factor. Afterall these items explicitly ask for both intrinsic and extrinsic motivation ("I study statistics to show myself and others that I'm ….."). Importantly, the CFA confirmed that it is better to consider items 5 and 12 as belonging to both the intrinsic and extrinsic factors. Only when assigning these items to both factors (and allowing them to correlate, as suggested by the modification indices) the CFA model fitted the data sufficiently. An overview of the fit indices of the CFA can be found in Table 3. The validity of the SMS is further confirmed by the correlations that were observed between the overall SMS score and statistics anxiety [$r(438) = -.23$, $p < .001$], math anxiety [$r(438) = -.19$, $p < .001$], hours of mathematics in high school [$r(438) = .19$, $p < .001$], and scores on the exam of statistics [$r(240) = .29$, $p < .001$] and general psychology [$r(231) = .18$, $p = .001$]. Again, the alpha level wat set to $\alpha = .01$ to correct for multiple testing and the 95% bootstrap confidence intervals were reported because the assumption of bivariate normality was violated for the correlation between the SMS score and the hours of mathematics variable [Shapiro Wilk's < .05]. All these correlations were significant and in the expected direction. An overview of these correlations can be found in Table 4.

Finally, the internal consistency of the average SMS score and the different subscales was determined by calculating Cronbach's Alpha. For the calculation of the subscales, items 5 and 12 were included in both the intrinsic and extrinsic scale. The Alpha's were .87, .90, .72 and .85 for the average score, intrinsic motivation, extrinsic motivation and amotivation subscales respectively. Taken together, it can be concluded that, although maybe item 5 and 12 could be revised, the current translation of the SMS has appropriate psychometric properties to be used in a Flemish population.

Q2 : What is the Level of Statistics Anxiety in Flanders?

Since no clear cut-off value to define SAS as a diagnostic category has been reported in the literature, prevalence cannot be quantified. For this reason, we provide the descriptive statistics including the average, the first quartile, the median and the third quartile. As can be seen in Table 5, both average and the median score on the SAS are just above 3, meaning that more than 50% of the students rate their statistics anxiety above the middle of the scale. When zooming in on the different subscales, it becomes clear that the "Examination Anxiety" pulls the average SAS scores, with an average of 4.00 and a median of 4.13. The scores on the other subscales where clearly lower. A repeated measures ANOVA with the three subscales as factor, showed a significant effect *[$F(1,79, 782.35) = 702.85$, $p < .001$]*. Post-hoc tests showed that all scales significantly differed from each other *[all $t(874) > abs(9)$, p's $< .001$]*. An overview of the level of SA can be found in Table 5.

Q3: Are There Differences in Statistics Anxiety and

Table 4
Pearson's correlations for the SAS and the SMS

			n	Pearson's *r*	*P*	Lower 95% CI	Upper 95% CI
Statistics Anxiety	-	Math anxiety	438	0.68	< .001	0.62	0.73
	-	Hours of math	438	-0.18	< .001	-0.29	-0.08
	-	Score statistics	240	-0.17	= .001	-0.30	-0.04
	-	Motivation	438	-0.23	< .001	-0.31	-0.14
	-	Score general psychology	231	-0.12	0.06	-0.26	0.01
Statistics Motivation	-	Math anxiety	438	-0.19	< .001	-0.29	-0.09
	-	Hours of math	438	0.19	< .001	0.10	0.28
	-	Score statistics	240	0.29	< .001	0.18	0.41
	-	Score general psychology	231	0.18	=.001	0.05	0.30

Table 5
The level of Statistics Anxiety

	Average SA	Examination	Interpretation	Asking questions
Valid	438	438	438	438
Mean	3.04	4.00	2.35	2.78
Std. Deviation	0.62	0.73	0.74	0.99
Minimum	1.29	1.25	1.00	1.00
Maximum	4.67	5.00	4.63	5.00
25th percentile	2.64	3.63	1.88	2.00
50th percentile	3.08	4.13	2.38	2.88
75th percentile	3.50	4.50	2.88	3.50

The Motivation To Learn Statistics Between First Year Students of The Professional and Academic Bachelor's in Psychology?

Statistics anxiety

To get an idea about group differences in overall SA, an ANOVA[2] was conducted with the average score of the SAS as dependent and Group (professional or academic) and Test Moment (begin or end of the semester) as independent variables. The ANOVA did not reveal any main or interaction effect [all *F*'s (1,434) < 2.42, all *p*'s >.12, η^2 <.006], indicating that both groups were comparable in terms of general SA at both moments. A more fine-grained analyses, focusing on the different SAS-subscales (i.e., exam anxiety, interpretation anxiety, anxiety for asking questions) revealed another picture. A MANOVA with these subscales as dependent and Group and Test Moment as independent variables revealed a significant effect of Group [Wilks' Lambda = .95, *F*(3, 432) = 7.47, *p* <.001, η^2 = .049]. The associated univariate ANOVA's showed that the multivariate effect was due to group differences in the subscales "interpretation anxiety" [*F*(1, 434) = 10.02, *p* < .002, η^2 = .020] and "anxiety for asking help" [*F*(1, 434) = 4.55, *p* = .03, η^2 = .01]. Interestingly, where the professional bachelor students showed more "interpretation anxiety" (2.43, *SD* = 0.74 compared to academic bachelor students (2.21, *SD* = 0.71), the reversed pattern was found for the "anxiety for asking questions" index with professional bachelor students showing lower levels (2.70, *SD* = 0.99) compared to the academic bachelor students (2.91, *SD* = .99). The (multivariate) main effect of Test Moment [Wilks' Lambda = 1, *F*(3, 432) = 0.32, *p* =.81, η^2 = .001] and the interaction between Test Moment and Group failed to reach significance [Wilks' Lambda = .99, *F*(3, 432) = 0.91, *p* =.44, η^2 = .006]. So, taken together, where psychology students in the professional and academic bachelor do not differ from each other in general levels of SA, they show a different underlying pattern, with professional bachelor students showing higher levels of interpretation anxiety, and academic bachelor students showing higher levels of anxiety for asking questions. Importantly, the levels of anxiety and the underlying pattern did not change throughout the semester. A visual impression of differences in SA between the professional and academic bachelor students can be found in Figure 1.

Motivation to learn statistics

To get an idea about group differences in overall motivation to learn statistics, an ANOVA was conducted with the mean score of the SMS as dependent and Group (professional or academic) as independent variable. This analysis revealed significant main effects of Group [*F*(1,434) = 6.87, *p* <.01, η^2 = .02] and Test Moment [*F*(1,434) = 4.66, *p* = .03, η^2 = .011]. The interaction between both variables failed to reach significance [*F*(1,434) = .01, *p* =.94, η^2 < .001]. The main effect of Group indicates that academic bachelor students showed higher levels of overall motivation to learn statistics (3.61, *SD* = 0.99) compared to the professional bachelor students (3.36, *SD* = 0.97). The main effect of Test Moment indicates that motivation was lower at the end of semester, from 3.57 (*SD* = 0.99) at the beginning to 3.37 (*SD* = 0.97) at the end of the semester. A more fine-grained analysis (using MANOVA) focusing on the different SMS-subscales (i.e., extrinsic motivation, intrinsic motivation and amotivation) revealed a significant effect of the factor Group [Wilks' Lambda = .96, *F*(3, 432) = 6.64, *p* < .001, η^2 = .04]. The associated univariate ANOVAs showed that multivariate effect was due to group differences in the subscale "Intrinsic motivation" [*F*(1, 434) = 12.69, *p* <.001, η^2 = .03] for which the academic bachelor students showed higher levels (3.12, *SD* = 1.20) compared to the professional bachelor students (2.72, *SD* = 1.17). The main effect of Test Moment *[Wilks' Lambda = .99, F*(3, 432) = 1.87, *p* =.13, η^2 = .01] and the interaction between Test Moment and Group failed to reach significance [Wilks' Lambda = 1.00, *F*(3, 432) = 0.64, *p* =.59, η^2 < .01]. So, taken together, academic bachelor students show a higher overall motivation to learn statistics compared to professional bachelor students. This difference is driven by higher levels of intrinsic motivation. Furthermore, the overall motivation declines over the semester, and this in an equal degree in both student groups. A visual impression of differences in SA between the professional and academic BA students can be found in Figure 2.

Q4: Are SA and the Motivation to Learn Statistics Predictors for the Score on the Statistics Exam?

From the descriptive statistics reported above, it becomes clear that the scores on the statistics exam correlates (in the expected direction) with SA, the motivation to learn statistics, and the amount of mathematics someone had in the last year of high school. Importantly however, besides being (potentially) influenced by these factors, scores on the exam will also be influenced by general learning efficiency. For the current study, we considered the scores on the exam of general psychology a proxy for the learning efficiency. It is important to note that scores on the exam of general psychology were only available for the students of Thomas More (from both test moments). Since not all participants gave permission to retrieve their exam scores and/ or they didn't participate in both exams, we could only use the data of 176 students. To avoid the inclusion of too many variables and interactions, the mean scores of the SAS and the SMS (instead of the subscales) and their interaction were used.

To get an idea whether these variables can explain

Figure 1

Boxplots of the SAS, the get a visual impression of the differences between professional and academic bachelor students.

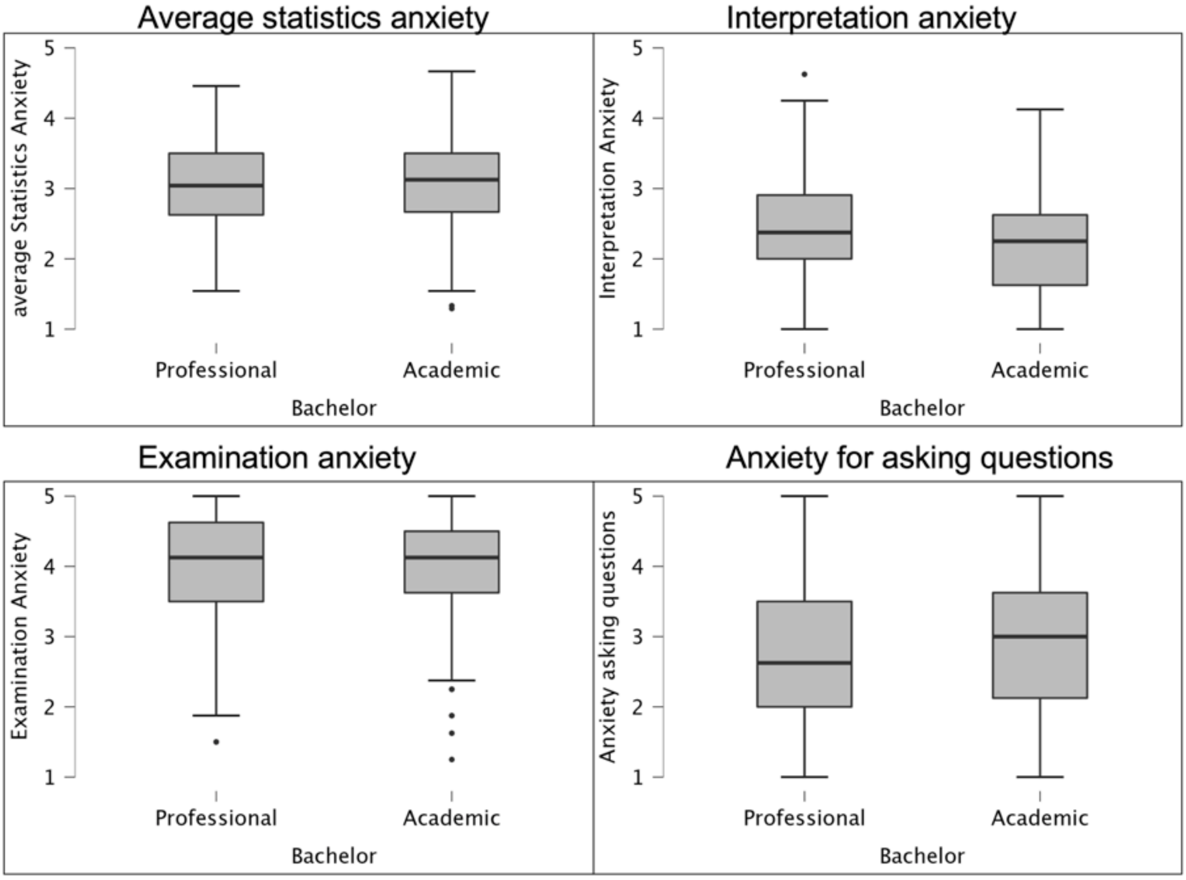
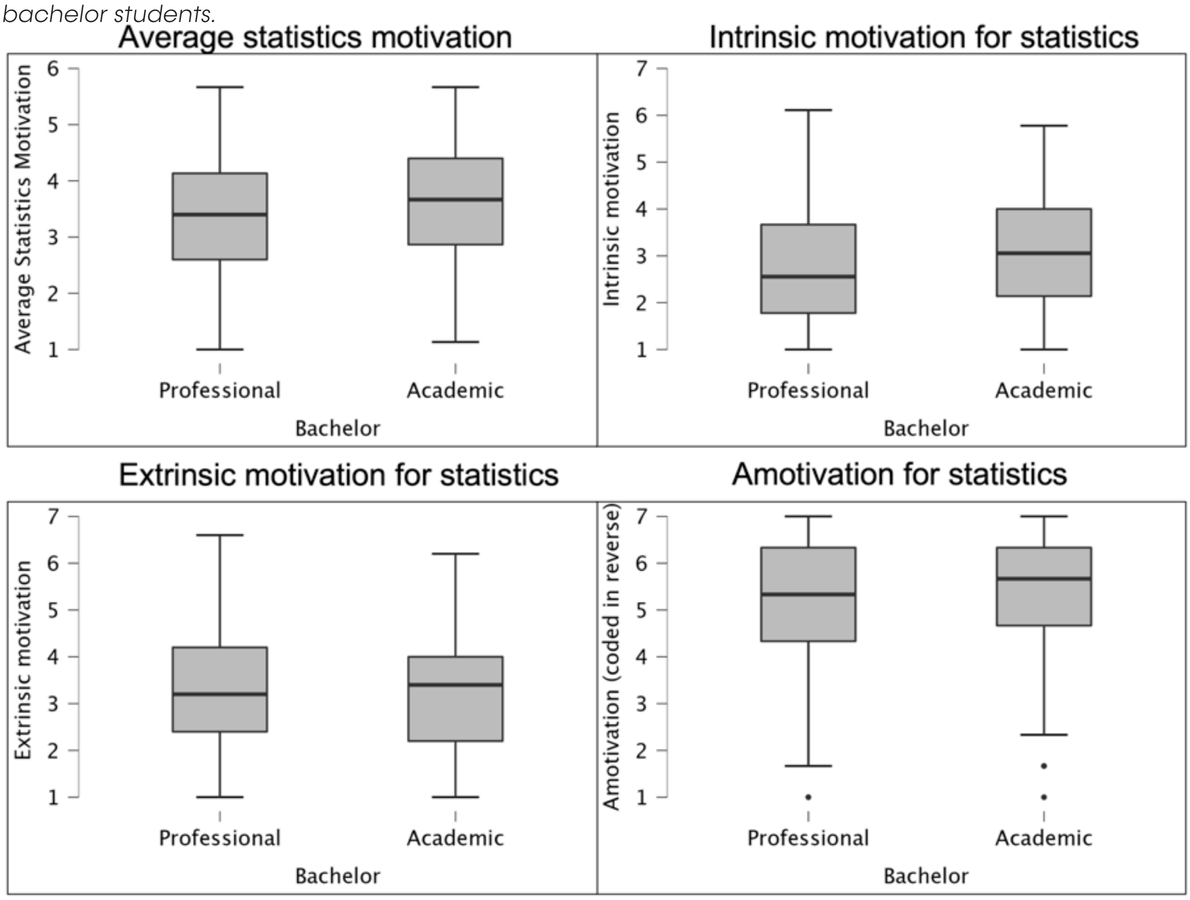


Figure 2

Boxplots of the SAS, the get a visual impression of the differences between professional and academic bachelor students.



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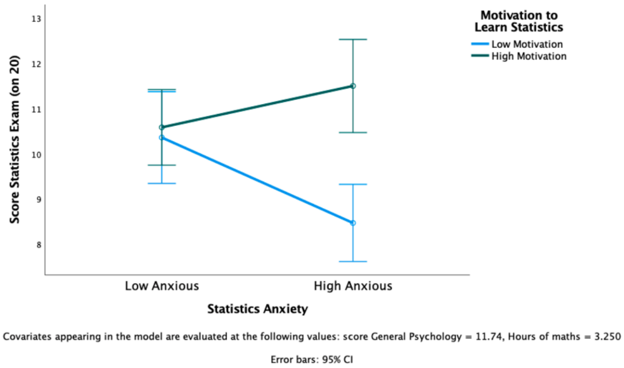
a part of the variance of the scores on the statistics exam, we conducted a (linear) regression analysis with the scores of the statistics exam as dependent variable, and the hours of mathematics in high school, the average SAS score, the average SMS score and the interaction between the SAS and SMS scores as independent variables. To control for this general learning efficiency, we added the scores of the exam of general psychology to the null model. This extended null model was significant [R^2 = .45, F(1, 174) = 142.33, p <.001]. More importantly, the R^2-change was significant [F(4, 170) = 6.41, p < .001], when the null model was compared to the model including hours of mathematics, the average SAS scores, the average SMS score and the interaction between SAS and SMS. This overall regression model was statistically significant [R^2 = .52, F(5, 170) = 37.14, p <.001]. It was found that the scores on general psychology significantly predicted the scores on the statistics exam [β(standardized) = .64, p <.001, partial = .67, semi-partial = .62]. The average SAS score was also a significant predictor [β(standardized) = -.58, p <.01, partial = -.22, semi-partial = .-16] as was the average SMS score [β(standardized) = -.59, p =.04, partial = -.16, semi-partial = .-11]. Finally, also the interaction between the average SAS and SMS scores was significant [β(standardized) = -.83, p <.001, partial = .21, semi-partial = .15]. Importantly, hours of mathematics was no significant predictor of the model [β = .10, p =.06, partial = .14, semi-partial correlation = .10]. Taken together, it can thus be concluded that, after controlling for general learning efficiency, SA, the motivation to learn statistics and the interaction between these variables are significant predictors for the scores on the statistics exam, together explaining a 7% of additional variance.

To be able to better understand the interaction between SA and the motivation to learn statistics, these variables were median-split and entered as factors in a full-factorial ANCOVA with general psychology and hours of mathematics as co-variates[3]. The pattern of results (see Figure 3) was virtually identical compared to the linear regression, with the exception that now, hours of mathematics became a significant predictor [F(1,170) = 4.87, p = .03, η^2 = .03]. Importantly, the interaction between the dichotomized SAS and SMS was again significant [F(1,170) = 8.69, p < .01, η^2 = .05]. Bonferroni corrected post-hoc analyses revealed that the students with high levels of SA and a low motivation obtained lower scores on the statistics exam compared to all other groups [all (abs) t's > 2.81, all p's < .04]. The other groups did not differ from each other. It thus seems that high levels of motivation can alleviate the negative impact of SA on the exam of statistics. Summarized, when leveling students in terms of general learning abilities, psychological factors like SA, the motivation to learn statistics as well as their interaction become convincing predictors for someone's success on the statistics exam (probably more convincing compared to the hours of mathematics obtained in high school).

Figure 3

The interaction between statistics anxiety and the motivation to learn statistics in relation to the scores on the exam of statistics.



General Discussion

The aim the present study was fourfold. (1) Since no validated SA and statistics motivation instrument exist for Flanders (Belgium), we translated the Statistics Anxiety Scale (Vigil-Colet et al., 2008) into Dutch and translated and adapted the Academic Motivation Scale (Vallerand et al., 1992) to measure the motivation to learn statistics (hereafter called the Statistics Motivation Scale or SMS). Here we aimed to validate these instruments for students in Flanders. (2) Next, we aimed to evaluate the level and distribution of statistics anxiety (SA) and the motivation to learn statistics (SM) in psychology students in Flanders. Importantly, because in Flanders the bachelor programs in psychology are organized both at the academic level (at universities) and at the professional level (at universities of applied sciences), (3) we compared both groups in terms of the level of SA and SM. (4) Finally, we aimed to determine the extent to which SA and SM (and/ or their interaction) relates to the scores on the statistics exam (above and beyond the mathematical background of the students). Below we provide comprehensive discussions of our outcomes for each of the research aims separately, and end with an overarching summary.

RA1: What is the Validity and Reliability of the Dutch Translations of the SAS and SMS?

The exploratory and confirmatory analyses conducted on both questionnaires showed that the construct validity of the instruments is acceptable. For both instruments, a 3-factor structure with loading >.40 was confirmed explain 56% of variance for both (which is an appropriate proportion, Scherer et al., 1988). For the SAS, the pattern of loadings was identical to that observed in the original version of the task (Vigil-Colet et al., 2008), with a minor change, that in our model, the error-terms of two items were allowed to correlate. As such, we could replicate the "examination anxiety", "interpretation anxiety" and "anxiety for asking questions" subscales. The correlated error-terms belonged to items that were associated with the "anxiety of asking questions"-factor and measured anxiety towards a private statistics teacher. Our guess is that the shared error-variance is related to the fact that not all students are familiar with a private teacher and had to use their imagination to answer this question. Interestingly, the same correlated error-terms were also observed by (Chiesi et al., 2011), in a validation of an Italian translation of the scale. In future attempts to improve the (Dutch translation of the) SAS, it could be considered to change or rephrase these questions. The construct validity of the SAS was further supported by the fact that we were able to replicate several previously reported findings. For example, in our sample we also found a strong and positive correlation between SA and math anxiety (Paechter et al., 2017). Importantly, the correlation was in terms of magnitude within the same range as those previously reported (r = .40 - .70), so that is can safely be concluded that also in Flanders, both constructs only partially overlap (only ca. 50% of shared variance). In addition, we also found a that subjects with higher levels of SA are overall less motivated to learn statistics (Dunn, 2014), and that students who had more hours of mathematics in high school also report lower levels of SA (Abd Hamid & Sulaiman, 2014; McGrath, 2014). Finally, we observed that females report higher levels of SA compared to males (Rodarte-Luna & Sherry, 2008). Importantly, the correlation between the SAS and the scores of the exam of general psychology was not significant, suggesting that the SAS measures something specific to statistics and not just test anxiety. In terms of reliability, the average SAS score and the scores on the different subscales had a high internal consistency.

The psychometric properties of the Statistics Motivation Scale were also acceptable. The SMS is an adapted version of the Academic Motivation Scale (Vallerand et al., 1992), and here we could largely replicate the factorial structure of the original scale. Except for two items, we found a similar loading pattern, and confirmed the presence of the "intrinsic", "extrinsic" and "amotivation" subscales. These two items originally loaded on the extrinsic subscale, while in our study they load on both the intrinsic and extrinsic subscales (and had correlated error-terms). Interestingly, this "double loading" is not so unexpected as both items start with "I study statistics to show myself and others that I…." and can thus be interpreted as both extrinsic and intrinsic. The validity of this scale is further supported by the correlations between the overall SMS score and statistics and math anxiety, hours of mathematics and the score on the statistics exam, which were significant and in the expected direction. Finally, also the internal consistency of the overall SMS and the different subscales was high.

Taken together, it can thus be concluded that the psychometric properties of both scales are good so that they can be used for future research efforts in Flanders, and that the scores on these scales can be directly compared to scores across other countries. At this moment, we don't have information about the test-retest reliability of both scales, as in the current study a between-subject design was used.

RA2: What is the Level of Statistics Anxiety in Flanders?

In our sample, many students report high levels of statistics anxiety as the average overall score on the SAS was 3.04 (on a scale from 1-5). This is especially the case for "examination anxiety" (EA: 4.00) of which the score was higher compared to the "Interpretation anxiety" (IA: 2.35) and the "anxiety for asking questions" (AAQ: 2.78) subscales. The scores obtained in Flanders, were comparable to those reported e.g. in Spain (Vigil-Colet et al., 2008; EA: 4.25; IA: 2.33; AAQ: 2.54), in Canada (Cantinotti et al., 2017; EA: 4.41, IA: 2.27, AAQ: 2.48), in Singapore and Australia (Chew & Dillon, 2014a; EA:4.12; IA: 2.64; AAQ: 2.54)), or in Brazil (Hernandez et al., 2015; EA= 3.62; IA:2.00; AAQ: 2.51).

The level of SA was similar in the beginning and at the end of the semester. So, in contrast to previous studies (e.g., Birenbaum & Eylath, 1994; Williams, 2013), we could not replicate the finding that general levels of statistics anxiety decrease when the statistics course progresses. It should be kept in mind, however, that our study was conducted in suboptimal circumstances to investigate the development of SA over time. First, due to restrictions imposed during the covid-pandemic, the classes switched to online (from which it is known that is can impact SA, DeVaney, 2010) teaching halfway the semester. Furthermore, the best way to investigate changes over time, is to use a within-subject design. We, however, used a between-subject design because this allowed us to obtain larger sample sizes. As such, it can't be ruled out that the stable levels of SA throughout the semester are due to different subsamples of students for whom the levels of SA change in the opposite direction. Future research (using a within-subject design) will be needed to shed further light on this issue.

Finally, although the levels of SA are undoubtably high (especially the examination anxiety) in our Flemish population, we don't know to what degree they are related to (general) trait anxiety or test anxiety. Although SA is related to trait/state anxiety, only a part of the variance in (the subcomponents of) SA can be explained by trait/state anxiety (correlations < .48, Walsh & Ugumba-Agwunobi, 2002). For this, it can be (carefully) suggested that in our sample the high levels of SA cannot be reduced to general levels of anxiety. Of course, to get a full picture of the mediating role of trait/ state anxiety in these numbers, it will be necessary to include such a measurement in future studies.

RA3: Are there Differences in Statistics Anxiety and the Motivation to Learn Statistics between First Year Students of the Professional and Academic Bachelor's in Psychology?

Above, we reported on the levels of SA when combining the data of the professional and academic bachelor students. When considering the overall levels of SA (taking the average SAS score), the professional bachelor students do not differ in SA compared to academic bachelor students. Interestingly however, the lack of group differences in the overall score are caused by opposite group differences at the level of the SA subscales. Where professional bachelor students show higher levels of "interpretation anxiety", they show a lower "anxiety for asking questions". Group differences were also visible in terms of motivation. Overall, the academic bachelors showed higher levels of motivation, and this was mainly due to higher levels of intrinsic motivation. These values of SA remain stable over time for both groups. The overall motivation to learn statistics, however, declined throughout the semester and this to a similar degree in both groups of students.

As mentioned in the introduction, interesting differences were found when comparing the levels of SA between countries (Baloğlu et al., 2011; Liu et al., 2011) or between groups with different (cultural) backgrounds within a country (Bell, 2008). These group differences are often attributed to differences in the educational system, different habits in using e.g., hand calculators or to the fact that students' native language differed from the languages used during instruction (Liu et al., 2011). In the current study, we show that meaningful difference can also be found between groups that share a similar study interest, have a comparable educational and cultural background, but who differ in the finality of their study program: the professional practice, or the academic master.

As mentioned above, the professional and academic bachelor programs in psychology differ from each other in several important situational and dispositional antecedents of SA. As mentioned in on the website "onderwijskiezer.be" (from the Student Guidance Centre (CLB), a center supported by the Flemish government, which provides future students with information for their choice of study), university colleges differ from universities in, among others, the distance between the students and the lecturer (which is typically smaller in the university colleges), in the way how the content of the courses fits the background knowledge obtained in high school (with a closer link at the university colleges), the size of the class groups (typically smaller at university colleges) and in the range of coaching and mentoring (with a larger and more personalized offer at university colleges). In addition, compared to the professional bachelor's in psychology, higher levels of mathematical background are recommended to enter the academic bachelor's in psychology program. Given these differences, is not unexpected that students in the professional bachelor have less anxiety to ask questions, while the academic bachelor students have less interpretation anxiety.

It is interesting to see that the levels of SA (the overall level and the subscales) do not change throughout the semester, and this for both groups. Indeed, this may be especially remarkable for the students of the professional bachelor, as it could be expected that the situational factors proper to the university college could work as a protecting or alleviating factor for the (development of) SA. It is important to note however, that the data-collection of the second wave took place during the covid-lockdown when all classes were organized online. This could have made the impact of the university college "atmosphere" less impactful. It would be interesting to investigate the evolution of SA during a "regular" academic year when all the classes were given on campus.

The situational and dispositional differences between the professional and academic bachelor programs can also be used to understand difference in terms of the motivation to learn statistics. Where the professional bachelor programs prepare students to become evidence-based professionals, academic bachelor programs typically attract students with stronger scientific interests. From this, it is again not surprising that the intrinsic motivation to learn statistics is higher in the academic bachelor students. Overall, the motivation to learn statistics decreased over the semester. Again, the abnormal semester (because of the covid-restrictions) makes it difficult to draw final conclusions. It would thus be interesting to investigate the evolution of SA during a "regular" academic year when all the classes were given on campus.

Taken together, we found meaningful differences between professional and academic bachelor students, both in terms of SA and in the motivation to learn statistics. Although the pattern of findings is not surprising, future efforts will be needed to pinpoint to the exact antecedents which are responsible for the difference. The current results are already important as they acknowledge that there exist important differences in the SA profile of different groups of students, and that for attempts to alleviate SA in their students, lecturers should consider the characteristics and background of their audience.

RA4 : Are SA and the Motivation to learn Statistics

Predictors for the Score on the Statistics Exam?

The relation between SA and the score on the exam of statistics has previously been investigated with mixed results. Where several studies reported a small but significant negative correlation between SA and statistical performance (typically ranging between r = -.20 and r = -.30; for an overview see, Macher et al., 2015) other studies demonstrated insignificant or even positive correlations (e.g., Lester, 2016; Paechter et al., 2017). One potential reason could be the lack of a proper control condition for general learning abilities. Afterall, the grades on a statistics exam are determined and influenced by a multitude of factors which are not limited to SA, the mathematical background, and the motivation to learn statistics. For this reason, we decided to investigate the relationship between these latter factors and the score on the statistics exam after controlling for these general leaning abilities. Here we operationalized these abilities by the scores on the exam of general psychology because this is also a course with a strong theoretical focus, but without the need for direct statistical calculation and interpretation.

We found, even when not controlling of these general learning abilities, a significant correlation between the scores on the statistics exam, the motivation to learn statistics and the hours of mathematics in high school. Interestingly, when introducing these factors and the interaction between SA and the motivation to learn statistics into a linear regression with the scores of statistics as dependent variable and when controlling for the general learning abilities. SA, the motivation to learn statistics, and their interaction remain significant predictors (explaining 7% of variance above and beyond the general learning ability), while the mathematical background failed to reach significance as a predictor. This interaction showed that SA only has a negative impact on those students who have a low motivation to study statistics. In other words, a proper motivation could act as a protecting factor for those students who experience high levels of SA.

Taken together, these findings are interesting because they add novel insights into the factors that predict someone's score on the statistics exam. Were previous studies pointed to the importance of a solid mathematical background for academic success in general, and the success on the exam of statistics in particular (Fonteyne et al., 2015), the current study indicates that it is also important to look at more psychological factors, especially when equating the students in terms of general learning ability. Given the high level of SA (see above) in psychology students, these are thus factors that should not be ignored by statistics lectures/ professors. Our study shows that motivation can act as a protecting factor to compensate for SA. Therefore, effort should be paid to (course/ class) interventions that can increase a student's motivation to learn statistics. In this context it is worrisome that in our sample, the motivation to learn statistics declined by the end of the semester (but this can be due to the covid-restrictions). Typically, anxiety and low motivation are faster and easier to tackle compared to an insufficient mathematical background. In fact, several tips to alleviate SA can be found in the list of situational factors listed in the introduction (like e.g. the pace of statistics instructions (Bell, 2005 as cited in Chew & Dillon, 2014b), the class atmosphere (Lesser & Reyes III, 2015), the absence of real-life examples (Neumann et al., 2013), instructor immediacy (Tonsing, 2018), the verbal and/ or non-verbal expressions of the lecturer (Williams, 2010), or the organization (online vs on campus) of the courses (DeVaney, 2010). When these interventions are not successful, a specifically dedicated training program to tackle SA can be considered, as several efficient programs to help student to get rid of/ or to deal with SA have been developed (Boaler et al., 2018; Smith & Capuzzi, 2019).

These tips and recommendations can not only be implemented by the statistics lecturer at the university or university college, but also by the math teacher in high school. We found high levels of SA in 1st year students already in the beginning of the academic year. This suggests that SA might have its roots earlier the school career of the students. In many countries, pupils are introduced to statistics during the math classes, and this already in the early years of high school (in Flanders e.g., the concepts average, median and modus are already discussed during the math classes in 2d grade, when the children are 12-13 years old). The results of our study indicate that, besides general mathematical background, (the interaction of) anxiety and motivation are significant predictors for statistics performance at the university or university college. Given the importance of statistics, it would be beneficial if the math teachers in high school would have an idea about the level of SA in their pupils and about their motivation to learn statistics. This would allow them to select and implement appropriate interventions to alleviate SA and to increase the motivation to learn it (if needed) early in the school career. The validated Dutch versions of the SAS and the SMS can be helpful tools in this context as they be used to detect the presence of SA and/ or to evaluate the impact of the implemented interventions. Afterall, it would be a big step forward if students can start their statistics courses in higher education with a good motivation and a healthy dose of self-confidence.

Summarized

In the current study we show that the Dutch translation of the Statistics Anxiety Scale (SAS) has good psychometric properties so that the obtained results can be properly interpreted and compared to the results obtained in other countries. In Flanders, psychology students report high levels of SA, with a level comparable to that observed in other countries. From the different SA subscales, the highest scores were observed for the "examination anxiety (EA)".

Although no differences were found in the general level of SA, professional and academic bachelor students differed in the degree of "interpretation anxiety (IA)" and "anxiety of asking questions (AAQ)". Where the professional bachelor students reported higher levels of IA, the academic bachelor students reported higher levels of AAQ. Importantly, these levels remained stable throughout the semester. The professional and academic bachelor students also differed from each other in their motivation to learn statistics, which the academic bachelor students showing higher levels of intrinsic motivation. Interestingly, the levels of motivation declined throughout the semester. This is unfortunate, because when controlling for general learning ability, the motivation to learn statistics interacts with SA when predicting someone's grades on the exam of statistics. A good motivation to learn statistics can alleviate the negative relation between SA and score on the exam of statistics.

Footnotes

[1] Interestingly, both questions are the only two asking for anxiety towards a private statistics teacher and were highly correlated (r = .84). Probably not all students had experience with a private teacher.

[2] Here and for the other ANOVA and MANOVA's, the Levine's test and/ or Box-*M* were never significant.

[3] These covariates are independent from both the SAS and SMS factors (an assumption of ANCOVA), as indicated by the lack of a significant differences between the high and low SAS and SMS groups for these covariates.

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Appendix A: the Dutch (translated) version of the AMS

Instructions in Dutch & (freely translated to) English (in Italic)

Denk bij volgende vragen telkens aan je ervaringen met het vak Statistiek 1 van je huidige opleiding.
(Think about your experiences with the Statistics 1 course of your current education.)

Gelieve aan te duiden in welke mate de volgende situaties je angst zouden bezorgen op een schaal van 1 (= geen angst) tot en met 5 (= heel veel angst).
(Please indicate to what extent the following situations would cause you anxiety on a scale from 1 (= no anxiety) to 5 (= a lot of anxiety).)

1. *Studeren voor een examen statistiek. (Studying for a statistics exam.)*
2. *Een tabel in een wetenschappelijk artikel interpreteren. (Interpreting a table in a scientific article.)*
3. *Individuele hulp vragen aan een docent statistiek voor de leerstof die ik niet zo goed begrijp. (Asking a statistics tutor for individual help with the subject matter I don't understand very well.)*
4. *De dag vóór mijn examen beseffen dat ik sommige oefeningen niet kan oplossen waarvan ik voordien dacht dat ze gemakkelijk zouden zijn. (Realizing the day before my exam that I cannot solve some exercises that I thought would be easy.)*
5. *Een privéleraar vragen om een statistisch onderwerp uit te leggen dat ik niet begrijp. (Asking a private teacher to explain a statistical subject that I do not understand.)*
6. *Een wetenschappelijk artikel lezen dat statistische analyses bevat. (Reading a scientific article that contains statistical analyses.)*
7. *Een docent vragen hoe ik een table voor kansrekenen moet gebruiken. (Asking a teacher how to use a probability table.)*
8. *Een wiskundig bewijs proberen te begrijpen. (Trying to understand a mathematical proof.)*
9. *Een examen statistiek afleggen. (Taking a statistics exam.)*
10. *Een advertentie over auto's lezen dat figuren en tabellen bevat over benzineverbruik en co2 uitstoot. (Reading an advertisement about cars that contains figures and tables about petrol consumption and co2 emissions.)*
11. *Het lokaal binnengaan om een statistiekexamen af te leggen. (Going into the classroom to complete a statistics exam.)*
12. *De docent om hulp vragen bij het maken van een oefening. (Asking the teacher for help with an exercise.)*
13. *De dag vóór het examen vaststellen dat je niet genoeg tijd had om de cursus te herhalen. (Finding out the day before the exam that you did not have enough time to rehearse the course.)*
14. *Wakker worden op de dag van het examen statistiek. (Waking up on the day of the statistics exam.)*
15. *Net voor de start van het examen statistiek beseffen dat je een bepaald type oefening niet hebt voorbereid. (Realizing just before the start of the statistics exam that you did not prepare a certain type of exercise.)*
16. *Een wiskundig bewijs overschrijven van het bord terwijl de docent dit bewijs aan het uitleggen is. (Copying a mathematical proof from the board while the teacher is explaining it.)*
17. *Een docent om hulp vragen bij het begrijpen van een afdruk met statistische resultaten. (Asking a teacher for help in understanding a printout with statistical results.)*
18. *Proberen om de kansen op winst bij een loterij te begrijpen. (Trying to understand the chances of winning in a lottery.)*
19. *Zien hoe een medestudent zorgvuldig de tabel met statistische resultaten bestudeert van een oefening die hij net heeft opgelost. (Watching a fellow student carefully study the table of statistical results of an exercise he has just solved.)*
20. *Naar een examen statistiek gaan zonder dat je genoeg tijd had om de leerstof te herhalen. (Going*

to a statistics exam without having had enough time to repeat the material.)

21. *Een docent om hulp vragen wanneer je een tabel met statistische resultaten probeert te interpreteren. (Asking a teacher for help when trying to interpret a table with statistical results.)*
22. *Statistische analyses proberen te begrijpen die in de samenvatting bovenaan een wetenschappelijk artikel staat. (Trying to understand statistical analysis that appears in the summary at the top of a scientific article.)*
23. *Naar het bureau van de docent gaan om vragen te stellen. (Going to the teacher's office to ask questions.)*
24. *Een privéleraar vragen om uitleg te geven over hoe je een bepaalde oefening maakt. (Asking a private teacher to explain how to do a certain exercise.)*

- Examination Anxiety Subscale: items 1, 4, 9, 11, 13, 14, 15, 20
- Asking for help Anxiety Subscale: items 3, 5, 7, 12, 17, 21, 23, 24
- Interpretation Anxiety Subscale: items 2, 6, 8, 10, 16, 18, 19, 22

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Appendix B: the Dutch (translated) version of the AMAS

Instructions in Dutch & (freely translated to) English

Denk bij volgende vragen telkens aan je ervaringen met het vak wiskunde tijdens het middelbaar.

For the following questions, always think about your experiences with the subject of mathematics during high school.

Gelieve aan te duiden in welke mate de volgende situaties je angst zouden bezorgen op een schaal van 1 (= geen angst) tot en met 5 (= heel veel angst).

(Please indicate to what extent the following situations would cause you anxiety on a scale of 1 (= no anxiety) to 5 (= very much anxiety).)

1. *Gebruik maken van de tabellen achteraan in het handboek wiskunde. (Using the tables at the end of the math textbook.)*
2. *Denken aan het wiskunde examen de dag voordat het examen plaatsvindt. (Thinking about the math exam the day before the exam takes place.)*
3. *Meevolgen hoe een leerkracht een wiskundig probleem uitwerkt op het bord. (Following along as a teacher works out a math problem on the black board.)*
4. *Een examen wiskunde afleggen. (Taking a math exam.)*
5. *Luisteren naar een wiskundeles. (Listening to a mathematics class.)*
6. *Een taak krijgen met veel moeilijke wiskunde oefeningen die tegen de volgende les moet ingediend worden. (Getting a task with lots of difficult math exercises to be submitted by the next class.)*
7. *Naar een medestudent luisteren die uitlegt hoe je een wiskundeprobleem oplost. (Listening to a fellow student explaining how to solve a math problem.)*
8. *Een quiz over wiskunde krijgen zonder dat je dit op voorhand wist. (Being given a pop-quiz on math without knowing it in advance.)*
9. *Een nieuw hoofdstuk in het wiskundeboek beginnen. (Starting a new chapter in the math book.)*

Appendix C: the Dutch (translated) version of the SMS

Instructions in Dutch & (freely translated to) English

Denk bij volgende vragen telkens aan je ervaringen met het vak Statistiek 1 van je huidige opleiding.

(Think about your experiences with the course Statistics 1 of your current program.)

Gelieve aan te duiden in welke mate onderstaande stellingen beschrijven waarom u statistiek studeert op een schaal van 1 (= helemaal niet) tot en met 7 (= helemaal).

(Please indicate to what extent the statements below describe why you study statistics on a scale from 1 (= not at all) to 7 (= completely).)

1. *Om later als ik afstudeer gemakkelijk een goedbetaalde job met aanzien te vinden. (To easily find a well-paying job with prestige, later when I graduate.)*
2. *Voor mij is statistiek zoals een spel: ik vind het leuk om te doen en ik leer nieuwe dingen bij. (For me, statistics is like a game: I like doing it and I learn new things.)*
3. *Ik kan niet begrijpen waarom we met statistiek moeten bezig zijn en eerlijk gezegd kan het me ook niet veel schelen. (I can't understand why we need to be involved with statistics and frankly I don't care much either.)*
4. *Omdat ik het leuk vind om goed te zijn in statistiek. (Because I like to be good at statistics.)*
5. *Om mezelf en anderen te bewijzen dat ik intelligent ben. (To prove to myself and others that I am intelligent.)*
6. *Voor de kick die ik ervaar wanneer ik interessante en uitdagende taken in statistiek oplos. (For the thrill I experience when I solve interesting and challenging tasks in statistics.)*
7. *Vroeger leek statistiek zinvol, maar nu vraag ik me af of het zin heeft om ermee verder te gaan. (Statistics used to seem useful, but now I wonder if it makes sense to continue with it.)*
8. *Eerlijk gezegd weet ik het niet. Soms heb ik het gevoel dat ik mijn tijd verspil met statistiek. (Honestly, I don't know. Sometimes I feel like I'm wasting my time with statistics.)*
9. *Omdat statistiek zal helpen om mijn droombaan te vinden. (Because statistics will help me find my dream job.)*
10. *Voor het plezier dat ik beleef wanneer ik mezelf overtref in statistiek. (For the pleasure I get when I outdo myself in statistics.)*
11. *Omdat statistiek me toelaat om veel nieuwe en interessante dingen te ontdekken. (Because statistics allows me to discover many new and interesting things.)*
12. *Op deze manier kan ik mezelf en anderen bewijzen dat ik statistische opdrachten zelf kan oplossen. (In this way I can prove to myself and others that I can solve statistical tasks on my own.)*
13. *Omdat als ik nu statistiek leer mijn kansen voor het vinden van een leuke job zullen vergroten. (Because if I learn statistics now, my chances of finding a nice job will increase.)*
14. *Omdat ik het leuk vind om statistische problemen op te lossen. (Because I like solving statistical problems.)*
15. *Omdat het bezig zijn met statistiek voor persoonlijke voldoening zorgt waardoor ik ook goed wil zijn in andere vakken. (Because being busy with statistics, gives me personal satisfaction which makes me want to be good in other subjects as well.)*

- Intrinsic Motivation Subscale (IM): items 2, 4, 5, 6, 10, 11, 12, 14, 15
- Extrinsic Motivation Subscale (ER): items 1,5, 9, 12, 13
- Amotivation Subscale (A): items 3, 7, 8

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An Instructional Design for The Improvement of Counting Skills in 3-Year-Old Children

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Abstract

The aim of this pre-experimental study is to evaluate the acquisition level of counting skills of a 3-year-old classroom made up of 14 children through a specific instructional design. To this end, an instructional proposal to improve these mathematical skills was designed. Before and after the intervention, we measured the students' level regarding counting skills through an evaluation of their counting abilities. The results indicate that the designed intervention increased the acquisition level of skills related to counting principles, constituting an effective instrument to enhance counting skills for 3-year-old children. In particular, after the intervention children improved significantly in skills related to the one-to-one correspondence principle and the order-irrelevance principle, both showing a large effect size in their observed differences. The cardinality principle, stable-order principle and abstraction principle also showed gains, but the differences were found to be statistically non-significant. Finally, the role of the age of the participants was also analyzed in relation to their acquired counting skills, indicating that children in the older age range improved their counting skills more than children in the younger group.

Keywords:

Early Childhood, Mathematics Education, Counting Skills, Counting Principles, Instructional Design

Introduction

In recent decades, mathematical learning has been a subject of study at both social and educational levels. The case of early counting skills, and its projection in the educational field, is notable since numerous studies have highlighted the relevant role that counting abilities play in typical and atypical cognitive development of early-childhood students (Baroody, 1992; Dowker, 2005; Gelman & Gallistel, 1978; Gelman & Meck, 1983; Hwang, 2020; Johnson et al., 2019; Rittle-Johnson & Siegler, 1998; Sarnecka & Carey, 2008; Wynn, 1992; Yilmaz, 2017). Mathematical educational research in pre-school stages reveals the importance of the acquisition of the necessary principles for counting (Baroody, 1992; Gelman & Gallistel, 1978). These principles have been identified as a base tool to start the learning of basic operations, as well as to establish a clear relationship between number and quantity. Despite the fact that the

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correct acquisition of these principles is necessary to properly perform counting-related tasks, from an adult point of view they may be considered extremely simple.

It is assumed that counting is a universal ability. All children over the age of 6 or 7 will count equally well, unless those with severe learning difficulties (Dowker, 2005). Research differs in how counting skills are acquired. Some researchers claim that counting principles master children's learning to count (Gelman & Gallistel, 1978; Gelman & Meck, 1983). Others state that the counting experience offers the child knowledge of counting skills (Fuson, 1988; Siegler, 1991). Yet others propose a mutual development whereby counting principles and counting procedures develop together during the learning of counting abilities (Baroody, 1992; Rittle-Johnson & Siegler, 1998).

Research has shown that students with learning mathematics disabilities and reading disabilities tend to show limited math abilities (Salihu & Räsänen, 2018). In particular, those children tend to show problems in understanding counting principles and detecting counting errors (Dowker, 2005;). For example, children with dyscalculia often demonstrate an incomplete understanding of some counting principles (Geary et al., 2000). Some other studies had shown that 6-years-old children who had difficulties with both mathematics and reading understand most of the counting principles, but consistently fail on tasks that assess order-irrelevance or consecutive count of contiguous objects (Geary et al., 1992; Geary et al., 2000). Studies centered on arithmetic abilities also showed that children with arithmetic disabilities in Grade 1 and 2 already had encountered problems on counting in kindergarten (Desoete & Grégoire, 2007; Stock et al., 2010).

Some studies have suggested that giving unselected children individual or small-group sessions of training in specific mathematical content and procedures, such as counting principles, comparison of quantities or quantity estimation can lead to significant improvement in typical and atypical children's number development (Ansari et al., 2003; Geary, 2011; Kaufmann et al., 2003; LeFevre et al.,2006; Stock et al., 2010). Instruction sequences based on counting tasks have also shown effects on individual differences in motor coordination, specifically related to the motor skills involved in using the fingers for counting (Dowker, 2005). In particular, in this study we present an instructional design aimed at favoring the typical development of counting skills for 3-year-old children.

The Acquisition of the Five Counting Principles for the Execution of Early Mathematical Action

The existence of different capacities or abilities that are key to learning mathematics from an early age have been a recurring subject of study for different relevant authors within the area of mathematics didactics and developmental psychology (Fuson, 1988; Sarama & Clements, 2009). One of these skills, which has been studied in depth from different perspectives, is counting. Counting is considered to be decisive for the progress of the cognitive and mathematical development of children (Fuson, 1988).

The work 'The child's understanding of number' published in 1978 by Gelman and Gallistel (1978) evidenced the existence of five counting principles that guide the acquisition of the ability to count, and allow the realization of a correct-counting process. According to this model, counting is made up of five principles namely: the one-to-one correspondence principle, the stable-order principle, the cardinality principle, the abstraction principle, and order-irrelevance principle.

The one-to-one correspondence principle. The one-to-one correspondence principle is defined as the assignment of a single number-word (Fuson, 1988) to each object in a collection. This skill involves the coordination of two processes: partitioning and labeling. On the one hand, as Lagos (1992) mentioned, the partition process is identified as the ability to divide the collection into two sub-collections: the elements counted and the elements that have not yet been counted. On the other hand, the labeling process refers to the ability of children to assign a numerical label to each of the objects that has been counted. Thus, Gelman and Gallistel (1978) consider that a child acquires the one-to-one correspondence principle when he/she is able to point to each element once, while assigning it a specific number-word. In addition, in relation to the principle of one-to-one correspondence, Briars and Secada (1988) identify three types of errors that occur in children's verbal productions when they count the elements of a collection. These errors are the omission of an object, the assignment of more than one number-word to the same object and the non-assignment of a number-word to an omitted object, even though this has been pointed out during the counting process. Regarding the age at which the one-to-one correspondence principle is acquired, Potter and Levy (1968) confirm that this skill is acquired at the age of two. However, many authors consider that this principle is not mastered before the age of 4 (Chamorro et al., 2005), and state that it is acquired by the age of 5 years (Briars & Siegler, 1984).

The stable-order principle. The acquisition of this principle is identified with the ability to count a collection repeatedly and assign the correct and conventional number-word to each item. For this principle to be acquired, the counting sequence has to be repeatable, this means that the number-words need to follow a stable and conventional order: a numerical sequence (Fuson, 1988). To be able to affirm that a child has assimilated the stable-order principle, it is necessary for the total number of elements of the collection to coincide with the stable and conventional part of the numerical sequence that the child masters. Fuson (1988), in his work, states that the average number of elements of the numerical sequence that 3-year-olds can recite adequately is five. Also, children around the age of four and a half are beginning to be able to recite a sequence of between 10 and 20 elements. Moreover, Chamorro et al. (2005) establish that learning the number sequence up to 10 corresponds to children aged four and a half years old, although this age is approximate because each child has individual characteristics and different learning rates. Finally, errors that are observed during the counting process in reference to the stable-order principle are called labeling errors. These are identified as errors derived from the action of labeling and are relative to the numerical sequence (Fuson, 1988). Geary et al. (1992) have shown that first-grade children with mathematics and reading disabilities understand stable-order principle.

The cardinality principle. The cardinality principle refers to the ability to designate the total number of elements in a set. In this way, the last number-word emitted when counting a collection has two different functions: to designate the last element of the collection and to determine the cardinality of the set. Gelman and Gallistel (1978) affirm that this counting principle is acquired at the moment in which the child repeats the last number of the counting sequence, or shows a special emphasis when pronouncing it aloud during the counting process. The acquisition of this counting principle occurs between four and five years old, considering that the acquisition of the principle implies having the ability to give cardinal meaning to the different numerical symbols (Chamorro et al., 2005). However, it should be mentioned that in order to acquire the cardinality principle it is necessary to firstly acquire the principles of one-to-one correspondence and stable-order (Gelman & Gallistel, 1978). As stated by Ansari et al (2003), the understanding of the cardinality principle is a good assessment on a proficient development of exact number representation and whether this follows a typical developmental trajectory.

The abstraction principle. This counting principle is defined as the ability to count a collection regardless of its qualitative aspect. In other words, it implies counting the elements of a collection without paying attention to concrete or abstract changes, such as alterations in the properties of objects, for example color or shape. In this way, children progressively understand that the properties of objects have no relevance during the counting process and, therefore, any object can be counted without influencing its qualitative characteristics. Regarding the age of acquisition of this counting principle, Gast (1957; cited in Gelman & Galistell, 1978) specifies that the age of acquisition of the abstraction principle in its entirety is identified at the age of seven.

The order-irrelevance principle. This last principle refers to the ability to count the elements of a collection without following any specific order. In other words, it is identified with the ability to understand that the total quantity of the set does not change regardless of the order in which its elements are counted. Chamorro et al. (2005) establish that the acquisition of the four previous counting principles is necessary to develop a numbered count. However, to affirm that a child is already capable of carrying out a total and correct counting process, it is necessary for the child to internalize the principle of order-irrelevance which allows the counting process to be given understanding and significance. Gelman and Gallistel (1978) affirm that the principle of irrelevance of order implies that the child is aware that the counted elements are independent of the label (number-word) assigned to them. This means that the number-word labels assigned during the counting process to each of the elements are assigned in a temporary and arbitrary way, and that the same cardinal of the collection is always obtained regardless of the order that is followed during the counting process. The principle of irrelevance of order, in the case of students with learning disabilities, is usually not fully understood (Geary et al., 2000).

Relevant Studies Regarding the Counting Principles

There are abundant studies regarding the counting principles in early school ages and, according to empirical evidence, there is a correlation between the acquisition of the counting principles and the success in which counting tasks are developed by typical children (Gelman & Gallistel, 1978; Gelman & Meck, 1983; Wynn, 1990; Lagos, 1992; Sarnecka & Carey, 2008; Johnson et al., 2019). Frequently, these investigations have tried to evaluate and analyze the performances of students based on different variables related to these tasks.

In the first place, with regard to the one-to-one correspondence principle, studies such as those by Gelman and Meck (1983) or Sarnecka and Carey (2008) evaluated via The one-one study and Correspondence task, respectively, the degree of acquisition of the one-to-one correspondence principle in pre-school children. Gelman and Meck (1983) presented a row of objects to be counted by children between three and four years old. After, the children had to detect any type of error in the counting carried out. The conclusion, and subsequent discussion of these tests, showed that children of both ages were able to identify the counting sequences where no type of error occurred and those in which errors were detected, although the percentage of children who detected errors was between 67% and 83%, versus 100% identifying the correct sequences. Likewise, in parallel to this study, Sarnecka and Carey (2008) evaluated the level of acquisition of the one-to-one correspondence principle in children from 2 to 4 years old. Within the analysis of the results, they highlighted that most of the children pointed to each of the objects and only assigned them a number-word. However, a previous study carried out by Gelman and Gallistel (1978) evaluated the one-to-one correspondence principle with the Videotape Counting Study task and concluded that two- and three-year-old children had almost no errors in the correspondence principle as long as the collections were made up of a reduced set of elements (between three and five). Thus, as the number of items increased, children began to show errors in this principle. Following the same line of research, Johnson et al. (2019), evaluated the level of acquisition of the one-to-one correspondence principle in pre-school children, through the tasks Count eight bears and Count 31 pennies. In their sample, 16% of the children were capable of developing this skill while 57.1% could only demonstrate the principle at certain times and, 26.9% had not yet acquired this counting principle.

Regarding the stable-order principle, Gelman and Gallistel (1978) developed the task The Magic Experiment. This task aimed to assess if young children differentiate between two categories of transformations performed on a collection of items. This study revealed that the majority of two-year-old children use a stable order sequence in spontaneous counting, in some cases with particularities. Three- and four-year-old children also use stable order sequences and made errors only in collections of larger sizes. Gelman and Meck (1983) concluded that the reason why the children presented problems with the stable-order principle was mainly due to the fact that they had not acquired the numerical sequence, in its entirety, following a stable and conventional order and, therefore, if they were asked to count a collection greater than the known numerical sequence, significant errors were found. In contrast, the study by Johnson et al. (2019), developed the Count 31 pennies and Count out loud tasks, the latter aimed to detect the highest number reached by a student using the standard sequence. A comparison between both tasks shows that slightly more children counted up to higher number when counting the pennies than when asked to count out loud without any objects (40.5% versus 35.5%).

Concerning the cardinal principle, the study by Gelman and Gallistel (1978) reported that the majority of four-year-old children were able to obtain, in most of the tests, the cardinal of the represented set. Gelman and Meck (1983), with The Cardinal Study, found that between 85% and 96% of three-year-old children already had an implicit knowledge of the cardinality principle. Wynn (1992) found that the mean age of the youngest children tested in their experiments was over 3-and-a-half, consistent with previous studies. A more recent work by Sarnecka and Carey (2008), reported that in 83% of the trials, the children, aged between two and four, answered the total amount of the collection adequately. In contrast, in 13% of the trials, the children answered incorrectly, and in the remaining 4% of the trials, they counted out loud instead of determining an exact final amount. The study by Johnson et al. (2019) provided a different analysis concerning the cardinal principle, as they studied the knowledge of the principle when engaging in more challenging tasks (counting 8 bears versus counting 31 pennies). Their results showed that, out of 317 children who provided a cardinal response to the bears task, 83 (26%) did not do so when counting the larger collection.

There are few references in relation to the acquisition of the abstraction principle and the order-irrelevance principle in early childhood education. Concerning abstraction, Markman (1979) and Fuson (1988) carried out counting experiments involving heterogeneous and homogeneous objects, indicating that by age 3 most children seemed to be able to take many different kinds of entities as separate equivalent "countable" units. Over the different studies reported in Fuson (1988), multiple count errors increased with object heterogeneity; however, children made more skim errors when the items counted were homogeneous (all the objects in a row were identical) than when they were heterogeneous. Concerning order-irrelevance, the dissertation by Lagos (1992) reported that the counting performance of 3 and 4-year-old children was significantly higher when few elements were presented in the counting collections and they were row-distributed instead of being arranged randomly.

Research Questions

The aim of this pre-experimental study is to evaluate the acquisition level of counting skills for 3-year-old children through a specific instructional design. Within this context, the research questions are as follows:

- RQ1: Is it possible to significantly increase the acquisition level of skills related to counting principles in 3-year-old students with an ad hoc designed intervention?

- RQ2: With this particular instructional design, are there significant improvements in any of the counting principles?
- RQ3: Is the difference in age a key factor for the improvement in counting principles in a 3-year-old Early Childhood Education classroom?

Method

In this study we follow a pre-experimental design with an ad hoc instructional sequence. We employed a pre-test/post-test design with the aim of assessing the children's improvement in counting skills before and after the intervention. The decision to not include a control group in this experiment was based on professional ethics, as we preferred an all-class intervention in order to maintain equity in the children's learning processes, which are highly influenced by what the teacher does in the classroom.

Participants

The study sample consists of a natural 3-year-old Early Childhood Education classroom in a public school in Valencia (Spain). A total of 14 children (8 boys and 6 girls) aged between 3 years 5 months and 4 years 4 months (M = 3.97) participated in the study. None of the children has special needs or a diagnosed learning disability.

Instrument

The measurement instrument was used to collect the data in both the pre-test and post-test with the aim of determining improvements in the acquisition of the students' counting principles after the intervention. To this end, the same evaluation method (rubrics) was applied to compare both measurements with a critical and objective character. In addition, the rubrics have served to determine the strengths and weaknesses of the design and the implementation of the instructional intervention. The rubrics were developed based on the studies of Gelman and Gallistel (1978) and Fuson (1988). Different rubrics were developed to measure the degree of acquisition of each of the counting principles.

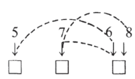
The results obtained by each student were coded by assigning specific scores. For each of the counting principles, the student's ability was evaluated over five consecutive attempts, each scored with 0.2. Thus, a total of five successful attempts results in the highest score (1) in that specific counting principle. In addition, since the five counting principles were evaluated and each one of them was assigned up to a score of 1, the sum of the test scores of each child was valued over 5 points.

The one-to-one correspondence principle was evaluated through the rubric presented in Table 1. The successful and failed counting attempts were defined, scoring the correct answers with 0.2 each. In addition, the type of error that the student made was determined according to the work of Briars and Secada (1988).

The evaluation of the stable-order principle was carried out in the same way, by counting the number of successful counting attempts. Following the study by Fuson (1988) regarding the learning of the numerical sequence during the acquisition phase, three significant parts in the structure of the counting productions of children were differentiated: the stable and conventional portion (an accurate number-word sequence), the stable but non-conventional portion (incorrect number-word sequence consistently produced) and the non-stable and non-conventional portion (incorrect number-word sequence that varied over trials). The rubric is shown in Table 2.

In the rubric referring to the cardinality principle (Table 3), three different categories were distinguished among the students' productions during the repetitions: i) cardinal not indicated; ii) cardinal indicated, but incorrect; and iii) correct cardinal indicated. Gelman and Gallistel (1978) consider that students have understood the cardinality principle as

Table 1
One-to-one correspondence principle rubric (add rows for each attempt)

ONE-TO-ONE CORRESPONDENCE PRINCIPLE					
Trial number	Correct trials	Trials with errors	Type of one-one error		
			Item omission	Assignment of more than one number-word	Non assignment of number-word
					

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long as they show behavioral manifestations, such as repeating the last element of the counting sequence, placing special emphasis on the last element of the counting sequence, or spontaneously repeating the last number-word once they finish counting. In addition, in the study by Wynn (1992) the responses of those students who are able to identify the last emitted number-word as the cardinal of the set are considered correct. Thus, the attempts in which the students indicate the cardinal are considered valid, regardless of whether the numerical result is correct or incorrect.

The abstraction principle rubric shown in Table 4 presents two variations in the counting items. Two changes have been evaluated in the properties of the collection objects: the color and the shape. Thus, five attempts were addressed varying the color, and five more attempts varying the shape. Regarding the scores, in both cases the trials were evaluated with 1, followed by an average of the results of both variations.

Finally, in Table 5 we show the rubric corresponding to the order-irrelevance principle. In each of the five counting attempts, the order of the elements was altered with random positions. To each of the trials, a score of 0.2 was assigned, giving a total score of 1.

Instructional Design

In order to answer the research questions, an intervention proposal was designed with a total of eight tasks devoted to the improvement in children' skills related to counting. The design of the tasks was based on manipulative materials as during the stage of Early Childhood Education, children begin to learn intuitive and informal mathematical knowledge based on exploration, experimentation, manipulation and situations involving play (Baroody, 1987). Moreover, all the tasks are based on common activities in school settings which they are likely to be familiar with. These tasks made up an instructional sequence to be implemented to the natural group of students. The details of the intervention are given in the following subsection. Below we give a brief description of each task:

Task 1: Put each cube in a recipient. To complete this task, the children must place a cube inside each container while they count out loud (Figure 1). Although different counting principles can be worked on with this activity, it is mainly oriented to improve the one-to-one correspondence principle, since the objective is to relate one cube with its container.

Table 2

Stable-order principle rubric (add rows for each attempt)

STABLE-ORDER PRINCIPLE					
Trial number	Correct trials	Trials with errors	Label errors		
			Stable and conventional portion	Stable but non-conventional portion	Non-stable and non-conventional portion
5					

Table 3

Cardinal principle rubric (add rows for each attempt)

CARDINALITY PRINCIPLE			
Trial number	Cardinal indicated, but incorrect	Correct cardinal	Cardinal not indicated

Table 4

Abstraction principle rubric (add rows for each attempt)

ABSTRACTION PRINCIPLE		
Trial number	Color variation	Shape variation
	YES / NO	YES / NO

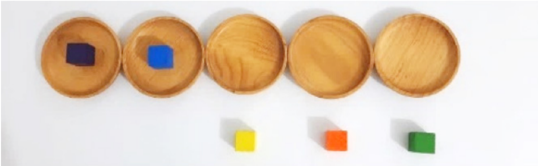
Table 5

Order-irrelevance principle rubric (add rows for each attempt)

ORDER-IRRELEVANCE PRINCIPLE	
Trial number	Order alteration
	YES / NO

Figure 1

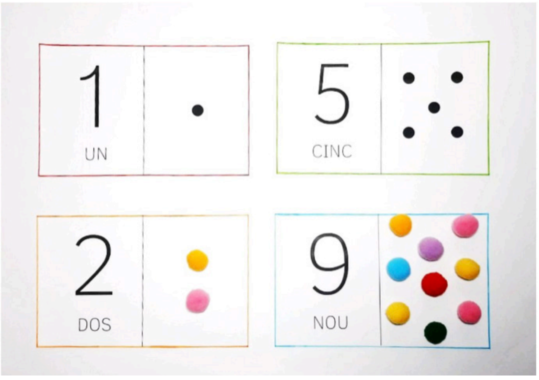
Put each cube in a recipient.



Task 2: Determine the quantity with modelling clay balls. A picture with a dual representation of a quantity is given on a sheet of paper, showing the number-word and the quantity in a black dotted-constellation format (see Figure 2, upper panels). The children have to assign a modelling clay ball to each of the depicted dots, while reciting the number sequence and determining the exact number of modelling clay balls they have to place, i.e., the cardinal of the set. This task is aimed at improving the one-to-one correspondence principle, the stable-order principle and the cardinality principle.

Figure 2

Determine the quantity with modelling clay balls



Task 3: How many are there? A series of cards with different numbers is distributed to a group of students. Then, the teacher sets up a collection of objects (feathers in the example shown in Figure 3). The children are asked to count the items out loud in order to determine the quantity. Once the total quantity has been indicated, the student who has the card with the correct number places it next to the collection. This task is mainly oriented to improve the stable-order principle, the cardinality principle and also the recognition of numerical symbols.

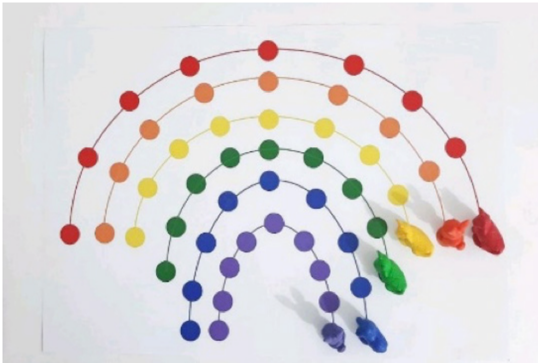
Figure 3

How many are there?

Task 4. Jump over the rainbow. The children are provided with the game board shown in Figure 4. The game shows a rainbow with different colored figures assigned to dotted paths over each rainbow color. The children are asked to move each figure from the starting point to the other rainbow side counting the jumps over the dots. After the counting process, the children are asked to state the total number of jumps that determine the cardinal of the collection for each color. Finally, the children are asked to start counting from the other side of the rainbow to move the figure to the starting position. The activity is focused on the one-to-one correspondence principle, the stable-order principle, the cardinality principle and the order-irrelevance principle.

Figure 4

Jump over the rainbow



Task 5. Roll, count and jump. The classroom floor is prepared with colored sheets of paper as depicted in Figure 5. The task consists of rolling the dice and counting the total number of dots on it. Once the children have determined the cardinal, they have to jump over the squares, while counting aloud, until they reach the corresponding number of colored sheets. The purpose of this task is to improve the one-to-one correspondence principle, the stable-order principle and the cardinality principle.

Figure 5

Roll, count and jump

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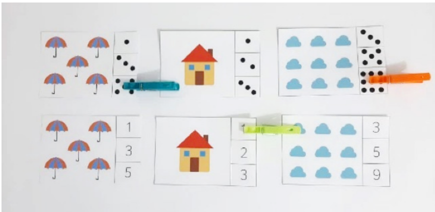
Task 6. Count and identify the cardinal. The task consists of counting the total number of elements and identifying them with the correct cardinal through different paper cards (Figure 6). As some students have difficulties with the recognition of numerical symbols, two models of cards are presented: those with numerical symbols (Figure 6, upper cards) and those with a dot-constellation representation (Figure 6, lower cards). This task aims to improve the one-to-one correspondence principle, the stable-order principle and, especially, the cardinality and order-irrelevance principles.

Figure 6

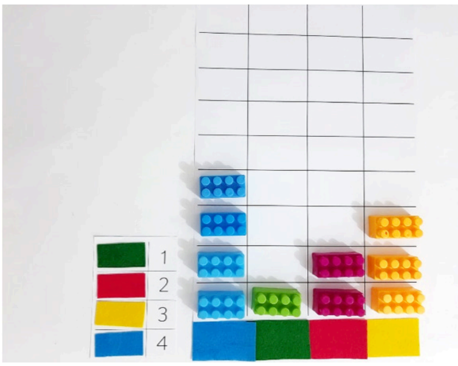
Count and identify the cardinal



Task 7. Count building blocks. The aim of this task is for the children to place building blocks on a grid (Figure 7, right), one piece in each square, as indicated by a card which has been previously given to each child (Figure 7, left), while counting out loud. This task is especially oriented to the one-to-one correspondence principle, the order-stable principle and the cardinality principle.

Figure 7

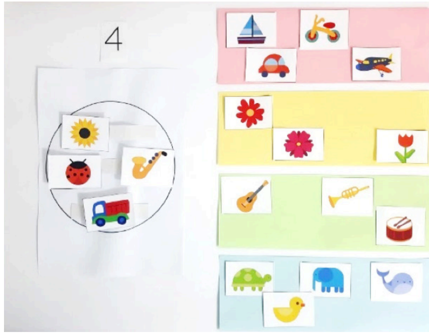
Count building blocks



Task 8. Create new collections. In this activity, the children are given different sets of items, classified into four categories: vehicles, flowers, instruments and animals (as depicted in the right panel of Figure 8). The children are asked to form a new collection made up of items from the given collections. The new collection has to be formed in the given circle (left panel of Figure 8) and, once the collection has been completed, the children have to count the total number of objects out loud and determine the final amount. Although all five principles are present in this activity, it is specifically focused on improving the abstraction principle.

Figure 8

Create new collections



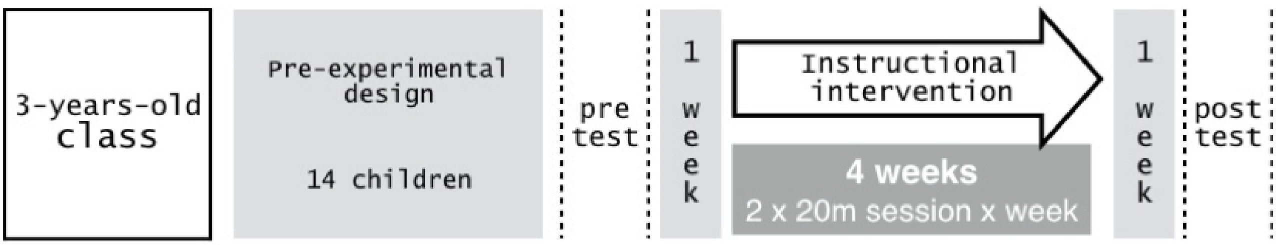
Procedure

In this pre-experimental design, all the children in the 3-year-old classroom were assessed with a pre-test for their counting-principle skills. The pre-test took place one week prior to the instructional intervention. After, the instructional sequence took place. One week after the end of the intervention the children were again assessed on their counting-principle skills through a post-test. Figure 9 gives a schematic view of the pre-experimental design. In what follows, we delve into the methodology carried out both in the pre/post-test and in the intervention.

Pre-test evaluation. The pre-test data collection was done by means of the instrument described above. An individual test, lasting approximately between ten and fifteen minutes, was completed by each child. To avoid unnecessary distractions, for each child the rubric assessment took place on a table away from the rest of the class. For each child, we first assessed, at the same time, the one-to-one correspondence, the stable-order and the cardinality principles with a total of five counting attempts, described as follows. A collection of nine blue rubber animal figures was presented with the intention of carrying out the counting sequences (Figure 10). The nine elements were provided in a linear arrangement in a horizontal format, and there was a space between each of the figures of approximately one centimeter. Each child was asked to count the entire collection out loud, pointing at each item as it was counted. Then, at the

Figure 9

Pre-experimental scheme



end of the counting process, the children were asked for the final quantity of the set by asking the question "How many animals are there?"

Figure 10

Collection for the pre-tests initial counting tasks

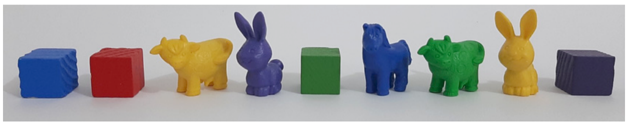
Secondly, the abstraction principle was evaluated with the two mentioned variants of the items: the alteration of color and shape. To get the children more involved, they were asked to choose the colors of the animals they wanted to count. So, the students lined up animals of different colors until they were told to stop, as shown in Figure 11. Afterwards, they were asked to count the collection in the same way as before (five attempts), out loud and pointing to the objects as they were counting them. Once the five counting attempts had been performed, four of the animals were exchanged for four different-colored cubes, as shown in Figure 12. The cubes were placed randomly within the horizontal objects' row and, therefore, sometimes there were cubes together or, on the contrary, they were interspersed forming a series with the rest of the animals.

Figure 11

Collection for the pre-test second counting task

Figure 12

Variant collection for the third counting task in the pre-test



Finally, to evaluate the order- irrelevance principle, a collection of nine blue animals was used again. Over five different attempts, the animals were randomly placed on the table in no particular order as shown in Figure 13. The children were asked to count the total number of animals out loud and point at them as they were counted.

Figure 13

Randomly placed figures for the last counting task of the pre-test

Instructional intervention. The instructional sequence was aimed to work with the children through the specially designed tasks. The methodology during the instructional intervention was based on learning corners, in order to favor the acquisition of mathematical knowledge (Clements et al., 2002; Sarama & Clements, 2003). In this way, the children were given more personalized attention and the opportunity to learn at their own pace. To this end, the classroom was divided into four groups (G1 to G4). Although this methodology is used for our specific classroom reality, the materials used in this proposal can also be adapted to other school realities where teaching resources do not allow the learning corners methodology.

Each day of the week (except Friday) two small groups carried out a 20-minute session devoted to completing one of the eight designed tasks. Hence, every two days a specific task was completed by all the children. Thus, it took four weeks to complete the instructional designed sequence, as shown in Table 6. The organization of the tasks in 20-minute sessions responds to the necessity of children's sustained attention during a short period of time. At age 4, it has been observed that children's attention span-persistence significantly predicted math achievement (McClelland et al., 2013).

Table 6

Task organization for implementing the instructional intervention

Week	Monday	Tuesday	Wednesday	Thursday
1	Task 1 G1 and G2	Task 1 G3 and G4	Task 2 G1 and G2	Task 2 G3 and G4
2	Task 3 G1 and G2	Task 3 G3 and G4	Task 4 G1 and G2	Task 4 G3 and G4
3	Task 5 G1 and G2	Task 5 G3 and G4	Task 6 G1 and G2	Task 6 G3 and G4
4	Task 7 G1 and G2	Task 7 G3 and G4	Task 8 G1 and G2	Task 8 G3 and G4

Post-test evaluation. One week after the instructional intervention finished, we collected the post-test data using the same measuring instrument used in the pre-test. Thus, the data collection was once again carried out individually, presenting the same counting activities to the children as described above.

Data Analysis

To address the research questions, we studied the differences obtained in the scores on the pre-test and post-test. Although the participating population is made up of only 14 students, the significance of the differences between the initial and final tests was determined using paired t-tests. To this end, the normality of the datasets was checked previously using the Saphiro-Wilk test (Saphiro & Wilk, 1965). The comparison of means was done using the standard t-student test for dependent samples in the case of normal distributed datasets. We used the Wilcoxon signed-ranked test (Wilcoxon, 1945), a non-parametric equivalent of the *t*-test, for non-normal data (Field et al., 2012). The analysis showed a significance level of 0.05. In the case where significant differences were found, the effect size r was calculated. A value of r = .10 means a small effect size, r = .30 can be classified as a medium effect size and r > .50 means a large effect size (Field et al., 2012).

Together with the mean score analysis we performed a detailed study on the success- per-counting attempt for each principle. These detailed studies were carried out in terms of the assessment-rubric attempts performed by the children both in the pre-test and post-test, as described in the following section.

Results

The results section followed a quantitative analysis on the gains in the children's counting skills, assessed from the pre-test and post-test. To explain the results in more detail, three subsections have been organized corresponding to the research questions initially posed. Thus, we will make the following distinctions: i) a global analysis regarding all five counting principles; ii) a specific and independent analysis on the evolution of each of the counting principles; and iii) an analysis concerning the improvement in the counting principles related to the ages of the participants.

Global Analysis

The obtained results in the pre-test ranged between 0.7 and 4.4, with 5 being the maximum possible score. The mean value of the pre-test scores was M = 3.14 (N = 14). As regards the post-test, the lowest score was 1.3 and the maximum score was 5 out of 5. The mean value for the post-test was M = 3.91 (N = 14). As shown in Figure 14 comparing both means, in general terms, a gain of 0.77 was achieved between the pre-test and post-test scores. Moreover, this difference between pre- and post-test scores is statistically significant with p = .0092, and r = .70, indicating a large effect size improvement.

Figure 14

Evolution of global counting skills assessed from the pre-test and post-test

Analysis of the evolution of each of the counting principles

Regarding each individual counting principle, the difference between the pre and post interventions shows gains in each of the counting principles, as shown in Table 7. In particular, the gain in the order-irrelevance principle (.23) was particularly noteworthy. The gain in the one-to-one correspondence principle (.21) and the gain in the cardinality principle (.20) are also remarkable. The rest of the counting principles also evolved, although with lower gains. Below, we discuss the significant differences encountered for each of the counting principles.

Regarding the one-to-one correspondence principle, the data from the pre-test and post-test was non-normal distributed, hence, the non-parametric Wilcoxon signed-ranked test was used for the contrast of means. The analysis revealed that the mean score obtained in the post-test (M = .90) was significantly higher than the mean score obtained in the pre-test (M = .69), p = .0069, with a large effect size r = .72.

In the case of the order-stable principle, both the pre-test and post-test scores followed a normal distribution. The dependent t-test reveals that the difference between the pre-test mean (M = .77) and the post-test mean (M = .80) was non-significant (p > .05).

The data obtained concerning the cardinality principle, the abstraction principle and the order-irrelevance principle in both the pre- and post-test assessments were non-normal for the three principles, thus the comparison of means was carried out again using the non-parametric test. As for the cardinality principle data, although finding differences between

Table 7.

Gains and scores obtained in the pre-test and post-test for each counting principle

	Counting principle										TOTAL	
	One-to-one correspondence		Stable-order		Cardinality		Abstraction		Order-irrelevance			
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Mean	.69	.90	.77	.80	.66	.86	.80	.89	.23	.46	3.14	3.91
Gain	.21		.03		.20		.09		.23		.77	

the pre-test score (*M* = .66) and the post-test score (*M* = .86), this was non-significant (*p* > .05). For the abstraction principle the results showed that the difference between the obtained mean in the pre-test (*M* = .80) and in the post-test (*M* = .89) was also non-significant (p > 0.05). Finally, for the order-irrelevance principle the Wilcoxon test showed a significant difference between the score obtained in the pre-test (*M* = .23) and the value obtained in the post-test (*M* = .46), *p* = .0042, *r* = .76, which represented a large effect size.

In the following, we report the results concerning the analysis based on the rate of -success attempts for each counting principle.

Detailed results on the one-to-one correspondence principle. Table 8 describes the evolution of the success rate per counting attempt for the one-to-one correspondence principle during the pre-test and post-test.

Table 8

Success rate per counting attempt on the one-to-one correspondence principle

Correct attempts	Pre-test	Post-test
5	28.57%	64.28%
4	14.29%	21.43%
3	35.71%	14.29%
2	14.29%	0.00%
1	7.14%	0.00%
0	0.00%	0.00%

A clear evolution and improvement can be seen in reference to the one-to-one correspondence principle after the intervention, since the total number of successful attempts was not less than 3 out of 5. In addition, during the post-test 64.28% of the children from the sample, equivalent to 9 students, managed to establish a correct one-to-one correspondence principle in each of the 5 attempts, compared to 28.57% of the children, 4 students, in relation to the pre-test.

Regarding the type of errors, three types of errors were evaluated during the counting attempts: item omission, assignment of more than one number-word and non-assignment of a number-word. As seen in Table 9, all three types of errors occurred on various occasions in both tests. Even so, there was a difference between the mean errors committed during the initial test (*M* = 1.57) and the final test (*M* = 0.5).

Table 9

Types of detected errors during the pre-test and the post-test

Student	Pre-test	Post-test
1	1 assignment error	0 errors
2	2 assignment errors 1 non-assignment error	1 assignment error 1 non-assignment error
3	1 item omission error 1 assignment error	0 errors
4	1 item omission error 1 assignment error	2 assignment errors
5	0 errors	0 errors
6	0 errors	0 errors
7	0 errors	0 errors
8	1 assignment error	0 errors
9	0 errors	0 errors
10	2 item omission errors 1 non-assignment error	1 item omission error
11	1 item omission error 1 assignment error	0 errors
12	1 item omission error 1 assignment error	1 item omission error
13	2 item omission errors 1 assignment error 1 non-assignment error	1 assignment error
14	2 assignment errors	0 errors
Mean error	1.57	.50

As can be seen in Table 10, during the pre-test the assignment of more than one number-word and item omission errors were more common. Regarding the post-test, although the number of errors was smaller, the error commission was still maintained. The distribution of error types was similar in both assessments: more assignment of more than number-word errors, followed by errors of item omission and, finally, errors of non-assignment of a number-word during the counting process.

Table 10

Absolute frequency of error types related to the one-to-one correspondence principle

Error type	Pre-test	Post-test
Item omission	8	2
Assignment of more than one number-word	11	4
Non-assignment of number-word	3	1
TOTAL	22	7

Detailed results on the stable-order principle. As shown in Table 11, there were slight differences between the success rates in the pre-test and post-test assessments. The results showed that 64.28% of the children achieved the maximum number of correct answers in both the pre-test and post-test. However, an improvement can be observed from the pre-test to the post-test, as the percentage of students who managed 4 correct attempts increased to 14.49% in the final test, compared to the initial test which was 7.14%. Even so, the rate of children who managed 2 correct attempts has reduced to 0.00% in the post-test, compared to 7.14% in the pre-test.

Table 11

Success rate per counting attempt on the stable-order principle

Correct attempts	Pre-test	Post-test
5	64.28%	64.28%
4	7.14%	14.29%
3	7.14%	7.14%
2	7.14%	0.00%
1	0.00%	0.00%
0	14.29%	14.29%

Regarding the type of errors detected during the evaluation of the order-stable principle (label errors), in general terms we found that, for our sample, in the stable and conventional counting sequence portion the children were able to count up to the number-word "five". Counting sequences with a stable, although non-conventional portion, were observed. Other cases with non-stable and unconventional portions were also detected. It was decided not to include any more information regarding this topic as it is out of the scope of the research questions posed.

Detailed results on the cardinality principle. During the pre-test, as specified in Table 12, the results showed that 64.28% of the children managed to establish the cardinality of the posed set in the five counting attempts. In addition, 7.14% made only one correct cardinality assignment and 28.58% were unsuccessful at all of the attempts. Regarding the post-test, the rate of children who acquired the cardinality principle in all five attempts increased to 85.71%, while 14.29% of the children were unable to correctly answer in any of the five attempts. Certainly, a clear evolution and improvement can be seen in reference to the cardinality principle after the intervention.

Table 12

Success rate per counting attempt on the cardinality principle

Correct attempts	Pre-test	Post-test
5	64.28%	85.71%
4	0.00%	0.00%
3	0.00%	0.00%
2	0.00%	0.00%
1	7.14%	0.00%
0	28.58%	14.29%

As mentioned in the instrument description, following the study by Wynn (1992), the attempts in which children identified the last emitted number-word as the cardinal of the set were considered correct. But in order to delve in the correct acquisition of counting skills, we analyzed the attempts in which the children indicate the correct or incorrect number-word. As can be seen in Table 13, the mean score of correct or incorrect last number-word emissions among all the students has been analyzed. Note that in Table 13 scores showing 0.00% in both correct and incorrect last number-word columns mean that the specific student failed in all counting attempts with regard to the cardinality principle. The results obtained show that children who emitted the correct last number-word in their counting attempts increased in the post-intervention assessment, as the mean score has increased from .43 to .76, showing an improvement in the general counting process.

Table 13

Correct and incorrect number-word emissions among the successful counting attempts performed by the children during the cardinality principle assessment

	Pre-test		Post-test	
Child	Last number-word correct	Last number-word incorrect	Last number-word correct	Last number-word incorrect
1	.80	.20	1.00	.00
2	.40	.60	.60	.40
3	.60	.40	1.00	.00
4	.60	.40	.60	.40
5	.00	.00	1.00	.00
6	.60	.40	1.00	.00
7	1.00	.00	.80	.20
8	.80	.20	1.00	.00
9	1.00	.00	1.00	.00
10	.00	.00	.80	.20
11	.60	.40	1.00	.00
12	.00	.20	.00	.00
13	.00	.00	.80	.20
14	.00	.00	.00	.00
Mean value	.43	.20	.76	.10

Detailed results on the abstraction principle. The results obtained from the alteration of color in the counting tasks in the pre-test and post-test showed no differences among counting attempts. Thus, in this subsection the abstraction principle will be specifically addressed concerning the shape variation in the collection to be counted over the attempts. As can be seen in Table 14, the success rates among attempts showed a slight variation from pre-test to post-test. This fact could be an indication that the abstraction principle is the one that is acquired later, as we will discuss.

Table 14

Success rate per counting attempt on the abstraction principle regarding shape-variations

Correct attempts	Pre-test	Post-test
5	57.15%	71.43%
4	0.00%	0.00%
3	0.00%	7.14%
2	7.14%	7.14%
1	0.00%	0.00%
0	35.71%	14.29%

Detailed results concerning the order-irrelevance principle. To conclude, in Table 15 we report the results concerning the success rate for the attempts on the order-irrelevance principle. As can be seen in Table 15, the children seemed to improve in correct attempts after the intervention. This is so because prior to the intervention the success rates were distributed between 0 to 3 correct attempts; after the intervention the percentages were more distributed, as 7.14% of the children did not succeed in any attempt, 35.71% succeeded in one attempt, and 14.29% succeeded in two and three attempts. Furthermore, unlike the pre-test, 21.43% of the children have four correct answers and 7.14% have five correct answers in four and five attempts, respectively.

Table 15

Success rate per counting attempt on the order-irrelevance principle

Correct attempts	Pre-test	Post-test
5	0.00%	7.14%
4	0.00%	21.43%
3	21.43%	14.29%
2	14.29%	14.29%
1	21.43%	35.71%
0	42.85%	7.14%

Analysis of the Influence of the Age Factor on the Scores Obtained

Finally, we report the results taking into account age as an analysis factor. To this end, the children were divided into two differentiated groups: Group 1 included children from 3.5 to 3.9 years-old; and Group 2 included children from 4 to 4.4 years-old. The mean scores obtained in the pre-test and post-test were again compared and analyzed based on the age factor. As can be seen in Table 16, the pre-test and post-test score comparison shows that the performance of the children in group 1 (3.5 to 3.9 years-old) was lower than the children in group 2 (4 to 4.4-old years), as the gain between the pre-test and post-test was 0.50 for group 1 and 1.03 for group 2. The mean scores were M = 3.1 in the pre-test and M = 3.6 in the post-test for the younger group, in contrast to the older group who scored M = 3.18 in the pre-test and M = 4.21 in the post-test. Statistical significance has not been determined for this age-separated sample, since the sample is too small and would lack statistical robustness, nevertheless, it seems that age is a key factor that affects the acquisition of counting skills.

Table 16

Mean scores and gain analyzed by age

	Age factor analysis			
	Group 1: 3.5 to 3.9 years-old		Group 2: 4 to 4.4 years-old	
	Pre-test	Post-test	Pre-test	Post-test
Mean score	3.10	3.60	3.18	4.21
Gain		.50		1.03

Discussion and Conclusions

This study has allowed us to explore the counting abilities of 3-year-old children to develop skills related to the counting principles, as well as to design an ad hoc intervention that improves these cognitive skills. This section discusses the results obtained in line with the bibliographic review proposed at the beginning of the paper in order to answer the research questions posed. We have arranged this section in terms of the research questions.

RQ1: Is it possible to significantly increase the acquisition level of skills related to counting principles in 3-year-old students with an ad hoc designed intervention?

To answer the first research question, an analysis of the mean scores obtained, both in pre-test and post-test, has been carried. Prior to the 4-week intervention, the children obtained an overall mean score lower than the average obtained after the intervention. This difference turns out to be statistically significant, indicating that the intervention was effective for the 3-year-old children, resulting in an improvement in the acquisition of counting principles. Also, the effect size of the intervention has been estimated as large.

RQ2: With this particular instructional design, are there significant improvements in any of the counting principles?

The results on the evolution of each of the counting principles, analyzed separately in the previous section, have shown that the five counting-principle skills improved after the designed classroom intervention.

In particular, concerning the one-to-one correspondence principle, the results showed that there is a statistically significant gain after the instruction. The performance of the children during the post-test showed that nearly 2/3 of the participants (64.28%) managed to establish the one-to-one correspondence when counting collections after the intervention, while less than 1/3 of the participants (28.57%) were able do this in the pre-test. Our findings are in line with Potter and Levy (1968), who affirm that the ability to establish one-to-one correspondence when counting collections is acquired at the age of two. In addition, our results align with the results obtained by other researchers (Gelman & Gallistel, 1978; Gelman & Meck, 1983) who state that children aged between three to four years-old are in the process of acquiring the one-to-one correspondence principle. In contrast, other authors report lower success rates in one-to-one correspondence counting skills. In Sarnecka and Carey (2008), children obtained an almost excellent global mean score in their one-to-one counting test. Similarly, Johnson et al. (2019) showed that less than 25% of their sample were able to successfully solve the one-to-one correspondence tasks. A detailed analysis concerning the errors related to the one-to-one correspondence principle has shown that all three types of errors were reduced after the intervention. As stated by Dowker (2005), children with low IQs are less good at number naming. They use to performed worse than their counterparts at detecting counting errors, especially with counting sequences beyond 5. From our study, although we have not carried out any IQ measurement, we concluded that the number of counting errors has been decreased after the intervention. In both assessments, prior to and after the intervention, the assignment of more than one number-word to an item when counting a collection was the most common error.

Skills related to the stable-order principle improved after the intervention, but with no statistical significance. Almost 2/3 of the participants managed to establish this principle over all the counting attempts in both tests. After the intervention, except for two children who had not yet acquired the stable-order principle, the rest of participants were able to establish the stable-order principle with a higher number of correct answers than errors. These results differ from those found by Johnson et al. (2019), who conclude that the percentage of students who could follow a stable and conventional order during counting tasks was less than 50%. Also, in contrast to the results obtained in our study, Sarnecka and Carey (2008) found that almost all the children in their study had already acquired the stable-order principle in its entirety. However, Fuson (1988) claims that 3-year-old students are already capable of using a stable and conventional sequence when counting up to five, while children of approximately four and a half years old are already beginning to be able to recite a stable and conventional sequence when counting between 10 and 20 elements. Chamorro et al. (2005) state that children are able to successfully count up to 10 with a stable and conventional number sequence at the age of four and a half years old. Thus, the results obtained in our sample, with children between 3.5 and 4.4 years old, align with Fuson (1988) and Chamorro et al. (2005).

Regarding the cardinality principle, the difference between the pre-test and the post-test scores was not significant. However, the percentage of children who improved their skills related to the cardinality principle increased after the intervention. In particular, 85.71% of the children established the cardinality of the set in all five counting attempts after the post-test. This finding aligns with the results of several authors (Gelman & Gallistel, 1978; Gelman & Meck, 1983; Wynn, 1992; Sarnecka & Carey, 2008), who state that children between 3 and 4 years old have already, broadly, acquired the cardinality principle. Moreover, the designed intervention has shown a positive effect on the rate of correct last number-words emitted during the counting attempts, as the rate of correct cardinality number-words increased after the intervention from .43 to .76.

The abstraction principle assessment showed a non-significant improvement after the instruction. The rate of children who correctly carried out the five counting attempts with varying shapes increased from 57.15% in the pre-test, to 71.43% in the post-test. Gast (1957) found the age of full acquisition of the abstraction principle to be seven years old. According to this claim, our post-test results report that 28.57% of the children have not yet fully acquired this counting principle.

The differences in the assessment of the order-irrelevance principle were statistically significant. Prior to the intervention, almost half of the children (42.85%) were unable to succeed in any of the five attempts. After the intervention, the rate of children who were unable to apply the order-irrelevance principle in any of the attempts decreased to 7.14%. Taking into account the success on the five performed counting attempts, this principle seems to be the one that was least strengthened by our designed intervention. This issue could be due to the fact that the order-irrelevance principle can only be fully acquired if the other four counting principles have been previously acquired, as stated by Chamorro et al. (2005).

RQ3: Is the difference in age a factor key for the improvement in counting skills in a 3-year-old Early Childhood Education classroom?

As observed, the mean scores after the intervention were influenced by the age of the children. Although both groups (3.5-3.9 years-old and 4-4.4 years-old) obtained very similar results in the pre-test, there was a considerable gain in the post-test scores for the oldest group. In fact, after the intervention the gain score was .50 for the younger group, meanwhile the older group obtained a gain score of 1.03. This result is aligned with the those of Gelman and Meck (1983), since, according to them, 3-year-old children show greater difficulty in identifying errors when performing a counting task compared to 4-year-old children. However, other studies, report that no significant differences are found between the counting skills of 3-year-old and 4-year-old children (Lagos, 1992).

Limitations and Final Remarks

Our study has shown the potential and the effect of an ad hoc intervention focused on improving counting abilities in 3-year-old children. Nevertheless, some limitations need to be underlined. First of all, the sample size is small as our study includes only 14 children, which could make the results not specifically representative. However, the diversity on the cognitive level observed among the participants, and the mathematics education literacy consulted, lead us to believe that a study with a larger sample will report similar results, although this claim should be confirmed by an experimental study. Moreover, this study was carried out in a real classroom scenario, in this way, the effects of these intervention could be exported to other school realities by other Early Childhood teachers. Another limitation is the pre-experimental design. The absence of a control group may pose problems regarding the intervention's effect validity on the level of acquisition of the counting principles. However, as has been argued in the methodology description, the pre-experimental design was intentionally chosen in order to offer the opportunity to the whole class-group to carry out the counting tasks and improve their cognitive abilities. The approach followed in this study was aimed at avoiding an imbalance in the class-group in relation to the counting process, as counting skills are a basic, essential part of elementary school practice.

Finally, despite that our intervention has been carried out with children without diagnosed learning difficulties, previous studies have shown the effectiveness of teaching sequences aimed at enhancing basic counting skills in both children who follow a typical or atypical developmental trajectory (Ansari et al., 2003; Kaufmann et al., 2003; Stock et al., 2010). With this in mind, the proposal that has been presented here and that has shown that it favors the acquisition of counting skills related to counting principles in typical 3-year-old students could also be useful as an effective instrument in students with learning difficulties.

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Using a Touch Point Instructional Package to Teach Subtraction Skills to German Elementary Students At-Risk for LD

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Abstract

This study replicates and extends prior research on multisensory mathematics instruction (Grünke et al., 2018) by integrating a touch points strategy, performance feedback, reward system, and a reinforcing card game into an instructional package. A multiple baseline design across participants was used to evaluate the effects of the touch points package on the subtraction skills of four German female first year students at-risk for learning disabilities. During intervention, the students were administered eight to eleven treatment sessions to learn how to subtract a one-digit subtrahend from a two-digit minuend up to 18 crossing over the tens barrier. Results indicated that all students made substantial increases in their subtraction performance during intervention. Student performance improved from 0 to 2 out of 10 math problems solved correctly during baseline to between 8 and 10 problems correct by the end of the intervention. Effect sizes observed across the four participants indicated the effectiveness of the intervention ranged from high to very high.

Keywords:

TouchMath, Subtraction Skills, Single-Case, Elementary, At-Risk, Learning Disabilities

Introduction

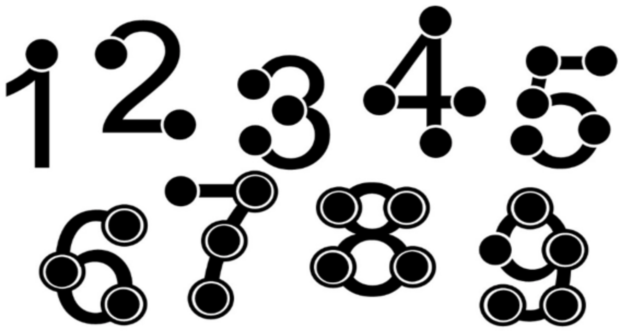
Mathematical difficulties are a pervasive problem for children and adolescents (Lein et al., 2020). About 17% of elementary and secondary students exhibit some form of mathematical difficulties and perform poorly, frequently well below school-grade expectations (Geary, 2015; Mazzocco & Vukovic, 2018). Many of these students, approximately 4%-7% of the school-age student population, will be later on identified with a learning disability (LD) in mathematics (Butterworth, 2019; Geary, 2011). Struggling with basic numeracy during the first school years are signs of mathematical difficulties (Stock et al., 2010; Tolaret al., 2016). Deficits in early math skills typically compound into further mathematical difficulties in the upper grade levels, and these difficulties often extend into adulthood (Bryant et al., 2020, 2021; Nelson & Powell, 2018; Powell & Driver, 2015; Powell et al., 2020). Thus, not addressing mathematical learning problems in the school context have negative long-term effects not only on the students' academic success, but also on later opportunities for vocational training, employment, and overall quality of life (Fischer et al., 2013; Geary, 2013; Kaufmann et al., 2020; Ritchie & Bates, 2013).

These concerns make particularly clear the imperative need to identify effective instructional practices (Mazzocco et al., 2018) that enable teachers to empower students' mathematical basic skills while helping them to overcome their difficulties at an early age (Dennis et al., 2016; Jitendra et al., 2018; Stevens et al., 2018). One such essential skill is the knowledge of number combinations (Kilpatrick et al., 2001). These are simple arithmetic problems (e.g., 4 + 5 = 9; 8 – 5 = 3) that can be solved by counting, applying decomposition strategies, or by automatic retrieval from long-term memory. The most fundamental way to meet this challenge is to use efficient counting strategies. While typically developing children do not need to be taught efficient counting techniques, students with mathematical difficulties do not discover them on their own (Ashcraft & Stazyk, 1981; Geary et al., 1987; Goldman et al., 1988; Groen & Parkman, 1972). Initially, learners use the strategy "counting all" to determine a sum. The application of more sophisticated counting strategies includes the possibility of "counting on" as well as their understanding of the commutative property of addition, which allows counting from the larger addend, regardless of the order they are in the arithmetic task (Baroody, 1995). The expansion of efficient counting and, later on, the acquisition of decomposition strategies, leads to more reliable retrieval of facts from working memory and a higher probability that these are also stored in long-term memory (Ashcraft & Stazyk, 1981; Geary et al., 1987; Goldman et al., 1988; Groen & Parkman, 1972).

Fuchs et al. (2010) demonstrated the effectiveness of a single intervention based on a direct instruction of the principles of strategic counting for students with arithmetic difficulties. In addition, they showed that when this intervention was combined with opportunities of deliberate practice, the positive effects on number combination became even more apparent. Furthermore, the authors recognized that effective instructional interventions as theirs and others in previous studies adhere to the following principles: (1) explicit instruction, (2) minimal learning challenge for students, (3) opportunity for practice, (4) and the use of motivators to help students with motivational and behavioral regulation (Fuchs et al., 2010, p. 98).

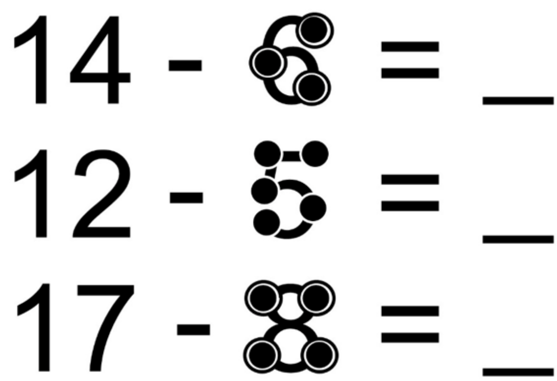
Another research-based intervention that aligns with the principles of Fuchs et al. (2010) is the use of touch points. This strategy fosters effective counting strategies and reinforces understanding of the cardinal concept of numbers in which the visual and haptic presentation of the quantity is used to associate it with the respective number word. The dot-notation approach, in which touch point dots are placed on the digits, was developed by Kramer and Krug (1973). Based on this method, Bullock, Pierce, and McClellan (1989) created the so-called TouchMath concept (see www.touchmath.com), an instructional mathematics curriculum to teach from basic addition and subtraction to more advanced arithmetic skills. Its central concept is that each number, according to its quantity, is illustrated with touch points. Whereas on numbers 1 to 5 shows only single points, on numbers 6 to 9, single and double points are used, accordingly (see Figure 1).

Figure 1
Numerals with Touch Points from 1 to 9



In the touch point strategy, students first learn to touch every point on a digit in a predetermined sequence while counting them aloud, which fosters cardinal number understanding. When solving a single-digit addition task, students use the counting-all strategy by tapping the dots on the summands. Thereafter, for addition tasks, learners are taught first to choose the bigger addend and, count forward while tapping the dots on the second addend. To solve single-digit subtrahend subtraction tasks, students are instructed to count backwards from the minuend tapping on the subtrahend touchpoints to reach the solution (see Figure 2 for some examples from a touch point subtraction worksheet).

Figure 2
Examples of Touch Point Subtraction Problems



Over time, touch points are faded from numbers to promote mental representations of quantities and acquisition of fact knowledge. The procedure of the touch point strategy is not limited to single-digit addition and subtraction; it can also be used for multiplication, division, and double-digit problems. One of the unique advantages of this method is its multisensory nature, which can be especially helpful for students with learning difficulties. Touching the points stimulates haptic perception, counting aloud activates auditory perception, and looking at the dots cues visual sensation (Scott, 1993).

Research on the touch points strategy has largely been conducted in the United States, some studies took place in Canada and Turkey, and one study in Germany (Grünke, Urton, & Karnes, 2018). Evaluation of the TouchMath strategy for students at-risk or with disabilities in mathematics instruction has focused mainly on addition, generally on single-digit addition (Ellingsen & Clinton, 2017). The investigations on the use of the TouchMath program were predominantly conducted at the elementary level, mostly with students with an intellectual or developmental disability, and very few targeted students with or at-risk for LD. However, there is a dearth of research on the effects of the touch points strategy to assists students with or at-risk of disabilities to acquire subtraction skills.

To date, only two studies were found that examinedthe efficacy of TouchMath for improving students with disabilities subtraction skills (Scott, 1993; Waters & Boon, 2011). Using a multiple probe design across four math skills, Scott (1993) assessed the effectiveness of TouchMath to teach three fourth grade students with disabilities, two with intellectual disabilities and one with LD, two-digit addition with regrouping, subtraction up to 18 with a single-digit minuend, and two-digit number with regrouping. The three students' performance on practice and novel problems was high after the introduction of the intervention for each of the math skills taught. In particular for subtraction skills, the students' performance score increased less than 14% on average during pre-intervention probes for practice and novel problems to over 86% on average in post-intervention probes. In the Waters and Boon (2011) study, three high school students with mild intellectual disability, two of which were also diagnosed with autism, were taught to perform three-digit money subtraction with regrouping using the TouchMath program. The strategy was effective to improve the subtraction skills of the students, with average improvement on performance of 69% to 83% from baseline to intervention. Upon completion of the intervention, one student maintained the subtraction skills over approximately 5 weeks, while another student experienced a substantial gradual decline in performance during 20 days. These studies show promise that the touch point system can assist students with disabilities to learn basic subtraction skills. Given the limited research on the effects of the touch point strategy on subtraction skills for students with or at-risk of LD, more studies are needed to explore its effectiveness for this population.

The purpose of the present study is to replicate and extend a previous experiment by Grünke et al. (2018) to examine the use of a touch points intervention package to teach subtraction skills to four first graders at-risk for LD. This study was aimed to answer the following research questions:

1. What is the effectiveness of a touch point instructional package to solve subtraction problems within 18 for students at-risk for LD?
2. What are the students' attitudes towards the touch point intervention?

Method

Setting and Participants

The study took place in a resource classroom in an urban public school in North Rhine-Westphalia, Germany during the last weeks of the school year. Four female first grade students at-risk for LD with ages between 6 and 7 years old enrolled in the same class at the school served as participants in the study. According to the school's curriculum, students are expected to have mastered the concept of subtraction up to 20 (e.g., 15-8) by the end of first grade, which in Germany constitutes the first year of formal schooling. Prior to the start of the study, the students had received instruction on addition and subtraction up to 20 using traditional instructional methods, however, math class instruction during the duration of the study did not focus on either addition or subtraction skills.

The eligibility criteria to participate in this study required participant students to: (a) be able to count forward to 20 and backwards from 10, (b) be able to count with one-to-one correspondence up to 20, (c) be able to add fluently within 20, (d) perform below grade-level on subtraction as required by the school's curriculum for first graders, (e) consent to take part in the study, and (f) have a high level of school attendance over the last six months. Before beginning the study, the special education teacher administered an informal test designed according to the diagnosis and training sheets by Klauer (1994) to evaluate the addition and subtraction skills of the students in her classroom. Based on the students' assessment scores, a detailed analysis of their addition and subtraction performance in their mathematics workbooks, and attendance records, the teacher and the second author identified four students that met the inclusion criteria.

The first participant, Aylin, was 7.6 years old. Her parents were both from Turkey and Turkish was their primarily language spoken at home. Informal assessments indicated that Aylin was unable to solve basic subtraction problems. She also showed neither a cardinal nor ordinal understanding of numbers, and was also unable to represent quantities or order them in relation to each other. The teacher characterized her as insecure in her mathematical abilities, but she was eager to improve her math skills. Aylin often would get upset when she experienced any kind of failure and frequently cried if she did not succeed on solving a mathematics problem.

Blanka was 6.8 years old and born in the Congo to French-speaking parents. She started learning German when she entered preschool. Even though she had mastered her addition facts through 20, she was unable to solve subtraction problems. However, she had a fairly well-developed ordinal understanding of numbers and was able to verbalize the steps she used in solving different mathematical problems. Blanka did not ask for help whenever she experienced difficulties; instead, she just waited for teacher assistance.

Carla was born in Germany and was 7.1 years old. She was mainly raised by her Turkish grandmother. Carla started to learn German when she was three years old, but still had trouble understanding the language. She had received special language training at her school since she enrolled. Carla had satisfactory addition skills; however, her subtraction skills were lacking.

Dana was a 6.10 years old girl, and her first language was German. She was able to represent quantities up to 20 and describe the steps she took to arrive at answers to various problems in mathematics. Her addition skills were acceptable for a first year student; however, like Aylin, Blanka, and Carla, Dana exhibited poor subtraction skills. According to her teacher she appeared very motivated to work on her subtraction skills, but often would become impatient with herself.

Since all four of the participants were still in their first year of elementary school, they had not yet been officially diagnosed with a disability. However, all the available academic data on their learning aptitude suggested they will soon be identified with some type of learning disability. Furthermore, the German proficiency level of the three second-language participants was neither formally evaluated at the time. According to the classroom teacher, except for Carla, the German skills of the other two students did differ, although not substantially, from those classmates without an immigrant background.

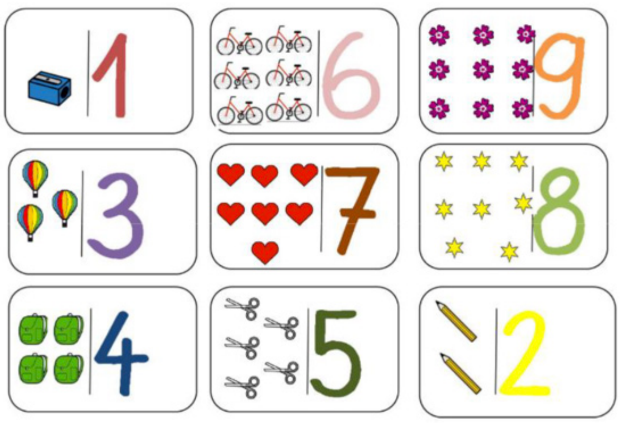
Two female special education graduate students served as interventionists in this study. Both interventionists were in their final months at the university before entering into the probationary teaching period to finish their training to become fully licensed special education teachers. Due to several months-long internships in schools and their jobs as teacher assistants, they both had ample experience working with struggling learners. To avoid conflicts with their teaching schedule at the school, the interventionists took turns to administer baseline and intervention sessions throughout the study.

Materials

Assessment materials included fourteen 10-item subtraction problems worksheets. Each subtraction problem consisted of a two-digit minuend up to 18 and one-digit subtrahend, where the tens had to be crossed to arrive at the correct difference (e.g., 12–8). The pool of subtraction problems meeting the aforementioned criteria were classified by two experienced first grade teachers in three levels of difficulty. All of the fourteen subtraction problems worksheets were designed to have a similar level of difficulty. A stopwatch was used to measure the time during assessment probes.

Intervention materials consisted of 4-inch numerals, dots, and minus signs made out of colorful sponge rubber, a set of cards, stickers, 10-item subtraction problems worksheets with dots in the subtrahend and worksheets without dots. All of the subtraction problems in the worksheets consisted of a two-digit minuend up to 18 and a single-digit subtrahend that required crossing the tens barrier. All worksheets had the same level of difficulty. A set of 1-inch by 2-inch laminated index cards displaying each a digit from 1 to 9 along with as many objects as the cardinality of the number. Finally, stickers with different motives based on characters from various popular cartoon series were used as rewards for performance during intervention.

Figure 3
Cards for the Card Game



Dependent Variable and Measurement

The number of subtraction problems solved correctly on a worksheet within 1-min was the dependent variable. Assessment worksheets were randomly administered without replacement to each student across baseline and intervention sessions. Two special education graduate students blind to the purpose of the study independently scored the mathematics worksheets. Inter-rater reliability was conducted on all of the assessment probes for each participant, and was 100%.

Experimental Design and Procedures

A multiple baseline across subjects design (Gast et al., 2018) was used to examine the effectiveness of the touch point instructional package to improve the subtraction skills of four elementary students at-risk for LD. The study was conducted over three weeks spanning across 14 sessions altogether for each participant. The intervention starting points were randomly assigned to the students to enhance the internal validity of the experiment (Tate et al., 2016), and staggered with baseline probes varying between three and six. Thus, the number of intervention sessions ranged from eight to eleven.

General Procedures. In each session, one of the interventionists took a student to the resource classroom and worked individually with the student for 20 minutes. After completion of the session, the student was given a 10-item subtraction problems assessment worksheet and was encouraged to work on the problems as fast and accurate as possible. Then, the interventionist started the stopwatch. After 1 min, the student was asked to stop working in the problems and praised for their effort.

Baseline. During baseline sessions, the student did not receive any instruction. Instead, the student and one of the interventionists worked together to make handicrafts. To prevent that differences in performance between the baseline and intervention condition might be due to an allocation effect, baseline sessions were set to last 20 minutes. After 20 minutes, a 1-min probe was administered to the student.

Intervention. The interventionists implemented a touch point instructional package that included: the use of the touch points strategy, performance feedback, performance-based rewards, and a card game. The intervention was comprised of six instructional lessons, each lesson taught within a session, followed by one or more independent practice sessions. At the beginning of each intervention session, the student was shown a chart displaying the number of subtraction problems she had correctly solved so far, and were told she would earn a sticker that could be placed on the chart if their performance was at least as good as in the previous session. Intervention sessions ended with a 5-min card game designed to reinforce learning of the touch point notation and counting as well. The card game was played as follows: First, a card was selected at random by the interventionist and given to the student, then the student stated the number of objects he saw on the card (e.g., two pencils, seven hearts) and lastly while touching the objects on the card he counted from the number up and then backwards. Following the end of the session, the student completed a 1-min assessment probe, after which, they received performance feedback, and were rewarded with a sticker if they maintained or improved their prior performance.

In the first lesson, the interventionist taught the student the touch points system using the sponge rubber digits and the dots, one digit at a time. The interventionist presented a rubber single-digit number displaying the appropriate touch points and then modeled how to count the touch points on a single-digit number. Next, the interventionist asked the student to practice placing the touch points on the rubber digit. Afterwards, the student named the number and then tapping on the touch points counted aloud forward from the digit up, following that, the student named the highest number reached and counted backwards while touching the touch points. For example, after placing the touch points on the digit 8, the student named the number 8 and then counted forward using the touch point from 9 to 16; next, he named the number 16 and immediately counted backwards down to 8 while tapping on the touchpoints. The student needed to perform each of these steps correctly on the digit before moving to the next digit. If the student made a mistake, the interventionist corrected the error and prompted her to continue. Three to four rounds of this procedure were required across all the students to learn the touch points on the rubber digits and count forward and backward correctly.

In the second lesson, several subtraction problems presented with rubber digits and dots were used to teach the student the touch point strategy to solve subtraction problems. The subtraction problems consisted of a two-digit minuend without touch points, and a one-digit subtrahend with touch points. To start the lesson, the interventionist showed a subtraction problem (e.g., 13-5) and proceeded to model the steps to solve the problem as follows: First, she started by naming the minuend (13) and then counted down accordingly to the number of dots on the subtrahend to reach the correct solution (12, 11, 10, 9, 8). Then, the interventionist demonstrated the procedure one more time with a second subtraction problem. Next, the student was instructed to solve a different subtraction problem while verbalizing aloud the steps to reach the solution. If the student made a mistake, the interventionist corrected the error and encouraged

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the student to continue solving the problem. The student practiced the subtraction procedure on at least four more additional subtraction problems, as many as time permitted.

Lessons three and four mirrored lesson two. In the fifth lesson, the student worked on a subtraction problems worksheet that displayed touch points on the subtrahends (see Figure 2). The student was asked to state aloud the steps she applied to solve each of the problems. If she had difficulties solving a problem, assistance was given by the interventionist, as needed. In the sixth lesson, the student worked on a subtraction problems worksheet without touch points. The student was instructed to draw the dots on the subtrahend before proceeding to solve a problem, and verbalize the steps to reach the solution. The interventionist constantly monitored the student's work on the practice worksheet. Help was provided when the student made an error when either drawing the touch points on the subtrahends or applying the steps to solve a problem.

During the independent practice sessions, the student was required to independently solve the subtraction problems on the worksheet that did not display touch points. The student was instructed to work through the problems to find the solution without drawing the touch points on the subtrahends. They received assistance if they asked for help from the interventionist.

Interventionist Training and Procedural Reliability

The interventionists received three 45-min training sessions conducted by the second author before the study began. Training on the procedures to teach the interventionists the touch point instructional package included explicit instruction, modeling, guided practice, and corrective feedback. In addition, the interventionists received training on the administration of the assessment probes. Baseline and intervention sessions followed a detailed step-by-step script to warrant a consistent implementation of the procedures. During each session, the interventionists marked on a checklist the steps they completed as they delivered the procedures. Both interventionists reported they completed each and all of the steps on the procedural checklists. Across all phases, the second author and the interventionists stayed constantly in contact by e-mail and phone to ensure the procedures were delivered as intended.

Social Validity

A teacher's assistant individually interviewed the four students after the intervention ended to capture their views and perceptions on the touch point instructional package. The student interview consisted of the following questions: (1) Did you enjoy calculating with the touch points? (2) Was calculating with the touch points easier for you than without them? (3) Would you prefer to continue calculating with touch points? And (4) How did you like getting constant feedback about your performance? Student answers were recorded, transcribed verbatim, and then analyzed in accordance with a simple approach outlined by Tesch (1990).

Data Analysis

The data analysis of the study includes visual analysis and descriptive statistics for each of the students across phases (Lane & Gast, 2014). Level, trend and stability was estimated for each condition. The stability criterion was set to 80% of data points falling within +/- 20% of the median (Lane & Gast, 2014). Furthermore, two commonly used non-overlapping effect sizes were calculated for each participant: percentage of non-overlapping data (PND) and Tau-U, to measure the effects of the intervention.

PND summarizes the percentage of intervention scores that exceeds the most extreme baseline score in the therapeutic direction (Scruggs et al., 1987). Participants' PNDs were averaged to obtain an overall PND. A PND over 90% suggests the intervention was very effective, from 70% to less than 90% effective, from 50% to less than 70% questionable, and below 50% ineffective (Scruggs & Mastropieri, 1998). Tau-U is a non-parametric effect size that can be interpreted as the percentage of improvement from baseline to intervention (Parker & Vannest, 2009; Parker et al., 2011). Tau-U computes a measure of the non-overlap between baseline and intervention phases while taking into account the intervention phase trend and also can control for monotonic baseline trend (Parker et al., 2011). Tau-U values range from -1.0 to 1.0, where a Tau-U value greater than zero indicates that intervention scores tend to be higher than baseline scores. Tau-U computation proceeded according to the steps laid out by Vannest and Ninci (2015): (a) the baseline trend level was calculated (Tau-U $_{\text{trend A}}$), and (b) if a baseline trend at or above 0.2 in the expected direction of the intervention was detected, the Tau-U coefficient that accounts for baseline trend (Tau-U $_{\text{A vs B + trend B – trend A}}$) was computed, otherwise, Tau-U without baseline correction (Tau-U $_{\text{A vs B + trend B}}$) was calculated. An omnibus Tau-U was obtained to measure the overall effect of the intervention. The decision to use either a fixed or random effects model to obtain the omnibus Tau-U was based on the heterogeneity of the data. A Tau-U value of 0.20 was interpreted as a small intervention effect, greater than 0.20 and smaller than 0.60 moderate, greater than 0.60 and less than 0.80 large, and over 0.80 very large (Vannest & Ninci, 2015). Finally, a piecewise regression analysis (level 1 analysis) for each participant and a hierarchical piecewise linear-regression analysis on the aggregated data (level 2 analysis) were conducted using the Huitema and McKean model (Huitema & McKean, 2000) to provide an inferential statistical validation of the results. Both level 1 and level 2 regression analysis were conducted using the SCAN package for R by Wilbert (2018).

Results

Figure 4 displays the students' number of problems solved correctly during the baseline and intervention conditions. Table 1 presents descriptive statistics on the performance of the students to solve subtraction problems.

Figure 4

Number of Subtraction Problems Solved Correctly for Aylin, Blanka, Carla, and Dana

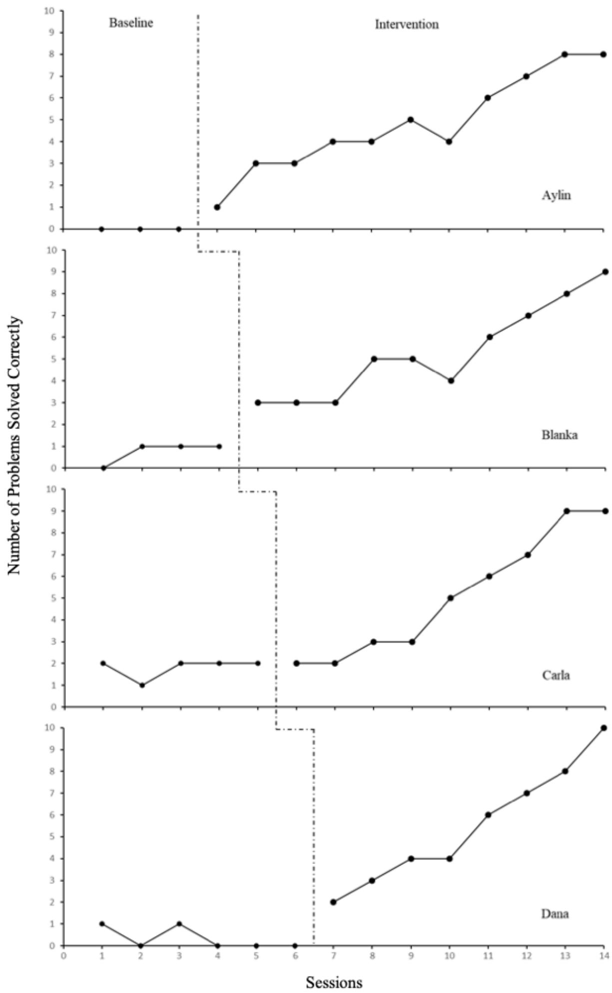

Table 1

Descriptive Statistics for Number of Subtraction Problems Solved Correctly

Student	*N*		*M (SD)*		*Md (IQR)*		Range	
	B	*I*	*B*	*I*	*B*	*I*	*B*	*I*
Aylin	3	11	0.00 (0.00)	4.82 (2.07)	0 (0.00)	4 (3.00)	0–0	1–8
Blanka	4	10	0.75 (0.50)	5.30 (2.16)	1 (0.25)	5 (3.50)	0–1	3–9
Carla	5	9	1.80 (0.45)	5.11 (2.80)	2 (0.00)	5 (4.00)	1–2	2–9
Dana	6	8	0.33 (0.52)	5.50 (2.73)	0 (0.75)	5 (3.50)	0-1	2–10

Note. B = Baseline, I = Intervention

Aylin. During the baseline phase, Aylin's performance was stable, she did not solve any of the subtraction problems correctly. In the intervention phase, Aylin demonstrated a stepwise improvement during the first six sessions with continuous improvement afterwards. Aylyn increased her performance from one subtraction problem solved correctly at the beginning of the intervention to eight by the last two days of the intervention. On average, Aylin solved 4.82 (range = 1 – 8) problems correctly during intervention.

Blanka. In the baseline phase, Blanka's performance was also low and exhibited a slight upward trend, with a mean of 0.75 (range = 0 – 1) problems solved correctly. Immediately after entering the intervention, her performance improved to three problems solved correctly within the first three intervention sessions, followed by a stepwise increase during the next two sessions, and then a steady growth in the last four intervention sessions. By the last intervention session, Blanka solved nine out of ten problems correctly. Blanka's mean performance during intervention was 5.30 (range = 3 – 9) problems solved correctly.

Carla. Carla solved mostly two problems correctly during the baseline phase. Her baseline performance was stable and averaged 1.80 (range = 1 – 2) problems correct. During intervention, Carla's performance started improving relative to baseline from the third session onwards, and continuously grew after the fifth intervention session. By the end of the intervention, Carla was able to solve nine problems correctly. On average, Carla solved 5.11 (range = 2 – 9) problems solved correctly during intervention.

Dana. In the baseline phase, Dana's performance exhibited a downward trend, she solved from zero to one problem correctly with a mean of 0.33 (range = 0 – 1). In the intervention phase, Dana increased her performance in a steady upward trend from two problems solved correctly just after the introduction of the intervention to ten by the end of the intervention, with a mean of 5.50 (range = 2 – 10) problems solved correctly.

The range of the effect size values suggests the touch point instructional package was effective to highly effective to improve the subtraction skills of elementary students at-risk for LD. In particular, PND across students ranged from 77.78% to 100%, with an overall PND of 94.45%. Tau-U effect sizes across the students ranged from 0.75 to 0.99 (p < 0.001), which are considered large to very large (Vannest & Ninci, 2015). Due to the lack of heterogeneity across the Tau-U effect sizes, a fixed effects model was applied to obtain an omnibus Tau-U. The overall Tau-U across students was 0.86 (CI95 = [0.66, 1.00], p < 0.01).

Table 2

Effect Sizes for Number of Subtraction Problems Solved Correctly

Student	PND	Tau-U [CI_{95}]
Aylin	100.00%	0.92** [0.52, 1.00]
Blanka	100.00%	0.81** [0.42, 1.00]
Carla	77.78%	0.75** [0.37, 1.00]
Dana	100.00%	0.99** [0.54, 1.00]
Omnibus	94.45%	0.86** [0.66, 1.00]

*Note. **p < 0.01*

Visual analysis indicated that baseline data was stable for two of the students, whereas one student displayed a slow decelerating trend and another a slight accelerating trend. An analysis of the students' baseline data determined none of the baseline trends were statistically significant. Thus, piecewise regression analysis of students' data and a hierarchical piecewise linear-regression analysis were conducted under the assumption of no baseline trend for all students. Due to the short duration of the baseline phases, this assumption theoretically may increase the risk of a beta error, which warrants a cautious interpretation of the results. As Table 3 illustrates, a statistically significant positive slope change estimate (Δ slope range = 0.65 – 1.10, $p < 0.001$) was found for all four students. On the other hand, a significant immediate change estimate with the introduction of the intervention was noted for three of the students (Δ level range = 1.33 – 1.59, $p < 0.05$), for one student the immediate change estimate was not significant (Δ level range = -0.69, $p = 0.15$). Moreover, visual analysis suggested that the performance growth of second language students was slower than the native language student, therefore, a hierarchical piecewise linear-regression analysis (level 2) was conducted to investigate the aggregated effect of the intervention and a potential interaction between intervention performance and second language learner status. Results showed a significant estimate for immediate change of 1.33 ($p < 0.05$) and a significant slope change estimate of 1.10 ($p < 0.001$) on the overall performance from baseline to intervention, however, no significant main effect for second language status (SLL = 0.58, $p = 0.44$) was observed. Furthermore, such analysis also revealed a significant slope change estimate (Δ Slope = -0.36, $p < 0.01$) from baseline to intervention between second language students and the native language student during intervention. This indicates that for second language students the performance slope during intervention was 0.36 slower than for the native language student. The overall immediate change estimate from baseline to the onset of the intervention for second language students was lower (Δ Level =-0.49, $p = 0.42$) than for the native language student, but this estimate was not statistically significant.

Table 3

Piecewise Regression Model for Number of Subtraction Problems Solved Correctly (Level 1 Analysis)

Student		ß	SE	t	Δ R^2
Aylin	Intercept	0.00	0.34	0.00	
	D Level[a]	1.59	0.48	3.35**	0.04
	D Slope[b]	0.65	0.06	11.50***	0.44
Blanka	Intercept	0.75	0.35	2.12	
	D Level[a]	1.52	0.55	2.79*	0.04
	D Slope[b]	0.67	0.08	8.63***	0.37
Carla	Intercept	1.80	0.26	6.95***	
	D Level[a]	-0.69	0.44	-1.57	0.01
	D Slope[b]	1.00	0.08	13.38***	0.61
Dana	Intercept	0.33	0.21	1.58	
	D Level[a]	1.33	0.40	3.37**	0.02
	D Slope[b]	1.10	0.08	13.70***	0.35

Note. a. Immediate change estimate from the baseline phase to the intervention phase.

b. Slope change estimate from the baseline phase to the intervention phase.

*Note. *p < 0.05, **p < 0.01, ***p < 0.001*

Table 4

Hierarchical Piecewise Regression Model for the Aggregated Student Data (Level 2 Analysis)

	ß	SE	df	t
Intercept	0.33	0.53	48	0.63
Δ Level[a]	1.33	0.53	48	2.49*
Δ Slope[b]	1.10	0.11	48	10.14***
sll	0.58	0.62	2	0.93
Δ Level[a]: SLL	-0.49	0.62	48	-0.8
Δ Slope[b]: SLL	-0.36	0.12	48	-3.05**

Note. SLL = Second language learner status.

Note. a. Immediate change estimate from the baseline phase to the intervention phase.

b. Slope change estimate from the baseline phase to the intervention phase.

*Note. *p < 0.05, **p < 0.01, ***p < 0.001*

Social Validity

End-of-intervention interview responses depicted positive students' attitudes towards the use of the touch points strategy. Aylin remarked, "I really liked working with the touch points. Even though I thought I would never learn how to subtract." She added, "The touch points made things very easy. I would like to use them in the future, too. Through working with the touch points, I lost my fear of math." Blanka explained, "Working with the sponge rubber digits was fun, and I really liked getting feedback on how well I did." However, when asked if she wanted to continue using touch points, she replied: "No, I don't need them anymore." Carla stated, "Being able to use the touch points made math very easy." She also voiced that she did not need the materials any longer: "I can do all the subtractions in my head now." She added, "Math is fun. I have not only become better, but also much faster." Dana said, "I liked the touch points, but not too much. Using my fingers is easier for me." However, she recognized, "I think that I can now do subtractions quicker and better." When asked if she would like to continue working with touch points, she answered, "No, I would rather play with other kids. I still don't like math very much."

Discussion

This replication study examined the effects of a touch point instructional package to foster the subtraction skills of four German elementary students at-risk for LD. Findings showed that the intervention was effective to very effective to enhance the ability of students to solve subtraction problems within 18 with two-digit minuends and one-digit subtrahends requiring crossing over the tens. All of the students were able to sustainably increase the number of correct responses using the touch point intervention from baseline (M = 0.72) to intervention (M = 5.18). Moreover, by the last two intervention sessions, students solved between 8 to 10 problems correctly as compared to between 0 to 2 problems during baseline. Students' performance improved during the course of the intervention as they learned and practiced the touch points strategy. Overall, both PND and Tau-U effect sizes at the individual and aggregated level were large to very large, which indicates the intervention was effective to very effective. Our findings are in alignment with previous research (Scott, 1993; Waters & Boon, 2011) that reported touch points instruction is effective to teach subtraction skills to students with disabilities.

Visual analysis in conjunction with a piecewise regression analysis indicated that the intervention did have a positive effect to improve the subtraction skills for all the students over time during the intervention. During intervention, there was an overall increase on the performance rate for all the students and an immediate increase in level from baseline to intervention for three of the students. It was noted that after the six instructional lessons, the performance across all the students continued improving in a steady manner. Thus, it is hypothesized that further independent practice upon completion of instruction helped the students to continue learning and internalizing the use of the strategy. In addition, hierarchical piecewise regression analysis results confirmed visual analysis that suggested that second language students' performance increased in a slower and more stepped fashion than for the native language student, this difference might have been due to some language struggles that these students might have had to overcome during instruction. However, this finding must be interpreted with caution as only one of the students was a native speaker. More interestingly, the second language students were able to catch up by the end of the intervention performing at the same level as the native language student. Thus, the hands-on and visual nature of the strategy along with direct instruction may have facilitated the acquisition of the steps to solve subtraction problems for all four participants. Furthermore, the results indicated that the use of touch points administered over a relatively short period, lasting from 8 to 11 sessions, yielded positive change on the performance across all the participants. This is consistent with previous results reported by Grünke et al. (2018), that found the same effects on learning single-digit addition skills of four German elementary students with intellectual and developmental disabilities. Therefore, findings from both studies suggests that providing a brief dosage of touch points instruction may be sufficient to effectively facilitate the learning of basic addition and subtraction skills for students with or at-risk for disabilities.

In terms of social validity, student responses to post-intervention interviews indicated that in general the touch points method was well-received by the students. Only one student, Dana, seemed not to be completely enthusiastic about using the touch points strategy. She stated the touch points procedure was more strenuous for her than finger counting. Unlike Dana, the other three students stated they enjoyed using the touch points strategy. Overall, by the completion of the study, all four students perceived an improvement of their subtraction skills, and felt the performance feedback provided during intervention motivated them to do better.

In summary, the findings of this study add to the growing body of literature on the effectiveness of the touch point strategy for students with disabilities. This investigation showed the touch point instructional package can be effective to improve the subtraction skills of first year German students at-risk for LD, some of which, were also second language learners.

Limitations

Several limitations should be considered in interpreting the results. First, the external validity and generalizability of the results is limited by the small number of participants. Further replications are needed to address this limitation. Second, participants were identified as at-risk for LD based on their academic performance during their first school year, and were selected according to the results of an informal mathematics assessment and an evaluation of their workbooks. Had standardized assessment data also been collected, it could have been determined whether the students met the criteria for an LD in Germany. Third, the intervention was conducted in a one-to-one format in a separate room. This instructional environment might have been conducive for strategy learning and thus boosting students' performance during intervention. Fourth, performance feedback and the use of rewards during the intervention may have contributed further to uplift students' motivation in learning the touch points strategy and performing better during intervention than in baseline. Even though these motivational factors could have been counter-balanced by the attention and encouragement delivered during the baseline phase, their influence on the students' intervention performance cannot be completely discarded. Fifth, repeated practice on the set of subtraction problems during the intervention sessions might have produced a facilitative effect that resulted in an overall increase of students' performance by the end of the intervention condition. Moreover, error correction procedures, performance feedback and repeated exposure to the set of problems during intervention might have promoted rote memorization of the answers. Sixth procedural reliability was self-collected by the interventionists. Due to interventionist bias, this method tends to inflate reliability ratings, thus weakening internal validity (Lane et al., 2009). In this study, however, the extensive interventionists' training and the use of detailed procedural scripts might have led to a reduction of the interventionists' bias and likely promoted higher procedural adherence and accuracy (Fallon, 2018; King-Sears, Walker, & Barry, 2018). Lastly, follow-up and maintenance data were not collected, thus, the short and long-term effects of the intervention are unknown.

Implications for the Classroom and Future Research

The findings of this study provide evidence that a touch point instructional package has the potential to enrich learning of subtraction skills for students at-risk for LD. However, to implement the method in practice, it is necessary that the teachers use differentiated instruction and adapt the instructional materials according to the competence level of each student. Additionally, because individual instruction in the schools is available in exceptional cases, future studies should explore the effectiveness of touch points instruction delivered in small groups or peer-tutoring formats in classroom settings. A peer-tutoring implementation of the touch point intervention seems to be promising as the strategy has a systematic approach that can easily be learned and conveyed by peer tutors. Future studies are warranted to evaluate whether touch point interventions are as effective to teach multiplication and division skills as well, and other math life skills such as money and time management. In addition, studies should also explore the effectiveness of the touch points method contrasted to other methods (e.g., number line) that enable students to expand their basic mathematics skills. Finally, future research should also investigate the effectiveness of computer-based instruction of the touch point strategy, such as TouchMath PRO, and other applications available within the program.

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Examining Elementary School Teachers' Perceptions of and Use of Formative Assessment in Mathematics

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Abstract

Formative assessment and related processes continue to prove to be a high-leverage instructional practice that has potential to support all learners, especially those who demonstrate misconceptions with significant mathematics concepts. Teachers use formative assessment practices in varied ways and share different perspectives of the value of these assessments for student learning. This article will share survey results of 65 teachers across grade levels. Findings indicate teachers find formative assessment beneficial for identifying gaps in learning, offers opportunity to increase student learning, and supports their teaching practices. These results support prior research; however, there were notable findings that offer insight into improving the use of formative assessment. The survey showed that formative assessment was used primarily to identify gaps, but not used to identify strengths of the learner. Formative assessment prompts focus on the learner but does not include reflection of the efficacy of the tool that was used or instruction. Commercially created materials, a large expense for schools, was not identified as useful. Teachers identified barriers to using formative assessment. Implications for improving formative assessment practices are shared and continued research.

Keywords:

Assessment, Differentiation, Digital Instruction, Elementary Education, Formative Assessment, Mathematics Education

Introduction

Learning Differences in Mathematics

In mathematics data from large-scale assessments continues to suggest that large gaps exist between students who are able to solve mathematical tasks and reason proficiently from those students who demonstrate opportunities to further develop and grow in their knowledge and understanding of mathematics topics (Institute for Educational Statistics, 2009; National Center for Educational Statistics, 2020; Organisation for Economic Cooperation and Development [OECD], 2019). On the large scale United States National Assessment of Educational

Progress (NAEP) assessment, fourth grade data has improved in general over the last 20 years, the gap between students' scores as well as the percentage of students who do not perform at the Proficient Level. For inclusion in this special issue on atypical learning in elementary school mathematics we provide an overview and findings of an investigation on teachers' use of formative assessment practices since formative assessment has been empirically shown to be a high-leverage teaching practice to support the mathematical development of all learners regardless of their abilities and backgrounds (Hattie, 2009; NCTM, 2014; Polly et al., 2016). While atypical learning and learning differences often eludes to children who have been identified as those with special or exceptional learning needs, we posit that all mathematics teachers should be adept and familiar with practices related to formative assessment which includes the process of assessing students, analyzing data, and determining subsequent instructional steps based on the data (NCTM, 2014; Polly et al., 2016; Polly et al., 2018).

Formative assessment is designed with the intent to understand the learner and use this understanding to provide instruction that is specific, but without a focus on ranking or ability grouping (McNeill & Polly, in press). Black and Wiliam (1998) highlight that the appropriate use of formative assessment is when the design and use culminate around student learning. Effective formative assessment practices include opportunities to use feedback, extend thinking, reveal reasoning, create goals, and engage in peer assessment (Baroudi, 2007; Black, Harrison, Lee, Marshall, & Wiliam, 2004; Heritage, 2007; Huinker & Freckmann, 2009; Polly et al., 2017). Formative assessment serves as a tool to address learning needs of students; however, these needs are not limited to challenges or misconceptions but rather the needs of the students holistically.

Background of Formative Assessment

The use of formative assessment has become common in classrooms as educators seek ways to use assessment data to differentiate instruction (Johnson, Sondergeld, & Walton, 2019). In their seminal work, Black and Wiliam (1998) describe formative assessment as teaching and learning activities that are adapted to meet student needs based on feedback received from students. Formative assessment supports the recursive feedback loop of instruction, assessment, analysis, and goal setting (Conderman & Hedin, 2012). The goal of formative assessment is to allow teachers to obtain systematic evidence about student thinking during instruction and to use those data to adjust and adapt instruction to meet individual students' needs. (Confrey, Toutkoushian, & Shah, 2019; Johnson, Sondergeld, & Walton, 2019; Wilson, 2018).

Formative assessments are typically informal and are embedded within an instructional activity. Examples include observations of students, student interviews or informal question-answer activities, admit slips or exit slips, journals, classroom discussions, and short written assignments (Bahr & Garcia, 2010). Technology tools, such as interactive white boards, mobile device apps, and educational software can support the use of formative assessment while providing students with immediate feedback (Pilli & Aksu, 2013).

By contrast, summative assessments are typically administered after instruction has occurred, with the goal of evaluating how well students have mastered the content or achieved the learning objectives (Bahr & Garcia, 2010). Summative assessments may take the form of a final exam, report card grades, or a large cumulative project. They may be used to evaluate school-wide goals or program effectiveness (Conderman & Hedin, 2012). Because they are administered at the end of a term or unit of study, summative assessments do not provide data that teachers can use during the learning process to adjust instruction (Garrison & Ehringhaus, 2007). Summative assessments are sometimes referred to as assessments of learning, while formative assessments are assessments for learning (Johnson, Sondergeld, & Walton, 2019).

Frameworks for Formative Assessment

Wiliam and Thompson (2007) suggest a formative assessment framework in which teachers implement the following practices:

- Explain to the students the learning objectives and the criteria for meeting those objectives.
- Facilitate effective discussions that provide students with opportunities to demonstrate their understanding of concepts and to ask questions about concepts that need further clarification.
- Provide ongoing feedback to students to advance their learning.
- Encourage students to serve as instructional resources for one another.
- Encourage students to take ownership of their learning.

Andersson and Palm (2017) expanded Wiliam and Thompson's framework to include three dimensions of the formative assessment process:

Dimension 1: Identify students' current understanding of the topic to be studied; identify the learning objective; develop a plan for moving students toward that objective.

Dimension 2: Establish the role of the teacher, peers, and learners in the formative assessment process. Keep in mind that all students are both learners and peers. The teacher may, for example, encourage students to serve as resources for one another and to monitor their own learning.

Dimension 3: Differentiate ways of implementing formative assessment in terms of the length and frequency of the formative assessment cycle. Teachers may consider how often instructional practices will be adjusted based on formative assessment data, as well as the amount of time taken to adjust instruction based on the formative assessment data (Anderson & Palm, 2017).

For example, short formative assessment cycles can occur within and between lessons, daily or weekly; medium formative assessment cycles can occur within or between instructional units (NCTM, 2007).

The Impact of Formative Assessment on Student Learning

Formative assessment correlates positively with student achievement (Andersson & Palm, 2017; Black & Wiliam, 1998; Furtak, et. Al., 2016; Hattie, 2009; Kingston & Bash, 2011). Klute, Apthorp, Harlacher, & Reale (2017) note that an analysis of 23 studies, all of which applied systematic, rigorous, scientific procedures, showed that students who participated in formative assessment performed better on measures of academic achievement than those who did not. Formative assessment used during mathematics instruction was found to have larger effects than formative assessment used during reading and writing instruction. In mathematics, both student-directed formative assessment and teacher-directed formative assessment were found to be effective.

Similarly, Yeh (2009) found a strong relationship between teachers' instructional adjustments based on formative assessment data and increased student achievement. Specific formative assessment strategies have been found to support student learning. Those strategies include peer-assisted learning (Rohrbeck et al., 2003), self-assessment using rubrics (Panadero & Jonsson, 2013), and self-regulated learning (Dignath & Buttner, 2008). Formative assessment was found to be more effective when teachers provided students with immediate feedback and made instructional adjustments early in the learning process based on formative assessment feedback. Early recognition of and response to learner needs through formative assessment analysis has been found to be important in preventing struggling elementary students from falling further behind their peers (Baumert et al., 2012; Conderman & Hedin, 2012).

While researchers broadly agree that formative assessment can promote student learning, more research is needed on specific formative assessments that are most effective (McMillian et al., 2013; Yan & Cheng, 2015). Dunn & Mulvenon (2009) note the difficulty in identifying best practices related to formative assessment, given the wide range of assessments available. This is particularly true for the application of formative assessments in mathematics education (van den Berg et al., 2018). Currently, there is much pressure on teachers to prepare students for high-stakes, summative assessment (Yan & Cheng, 2015). Formative assessment, then, tends to be viewed as an extraneous task, rather than as an integral part of teaching and learning (Coffey et al., 2011). Research is needed on how best to prepare teachers to implement effective formative assessment.

Research Questions

This study was guided by the following research questions:

RQ 1: What are elementary school teachers' descriptions of formative assessment in mathematics?

RQ 2: What benefit do elementary school teachers report about formative assessment in mathematics?

RQ 3: What barriers do elementary school teachers report related to formative assessment in mathematics?

RQ 4: What resources do teachers find useful for conducting formative assessment?

RQ 5: How does formative assessment help teachers differentiate mathematics instruction?

Methods

Participants and Procedures

To answer the research questions, we created an online survey using SurveyShare that included both Likert scale and open-ended items. Once the survey was created, we had the survey read by two elementary school teachers to make sure that the questions were clear and understandable.

Participants in this convenience sample were recruited to complete an online survey based on e-mail messages to the authors' current and former students as well as social media postings on Twitter and Facebook. Sixty-two participants completed the survey, 53 of whom identified themselves as elementary school teachers. Table 1 describes the grade level taught by the participants at the time that they completed the survey.

Table 1
Grade Level of Participants

Grade	Number of Participants
Kindergarten	3
Grade 1	6
Grade 2	9
Grade 3	6
Grade 4	9
Grade 5	20

Data Analysis

In order to answer the various research questions multiple processes were used in this mixed methods study (Johnson & Onwuegbuzie, 2004). Research question 1 was analyzed by a thematic open-coding process of participants' responses to the survey question, "Describe in a sentence what formative assessment means to you related to teaching mathematics." Responses were coded based on participants' response and then responses were sorted and organized by code.

Research question 2 was analyzed by calculating the percentage of participants that strongly agreed, agreed, disagreed, and strongly disagree to four statements on the survey. Percentages were also calculated for research question 4 to find what resources teachers found very useful, useful, somewhat useful, not useful, or not applicable. For research question 2, 3, and 5 open ended survey questions were also analyzed using a thematic open-coding process of participants' responses (Miles et al., 2019). Once the data was coded themes were generated. Those themes were then confirmed by revisiting the original open-ended survey responses.

Findings

RQ 1: What are elementary school teachers' descriptions of formative assessment in mathematics?

Table 2 provides the codes and frequencies of each of the codes related to the survey question, "Describe in a sentence what formative assessment means to you related to teaching mathematics." The most frequent codes from participants were that formative assessment was used to assess learning (35, 66.04%) and that it can provide an informal check for understanding (35, 66.04%). The assess learning code was primarily found in older grades and was mentioned by 77.14% of participants who teach in Grades 3-5 compared to only 44.44% of participants who teach in Grades K-2. The code that formative assessment provides an informal check for understanding was mentioned by 83.33% of the participants who teach in Grades K-2.

The analysis of participants' responses showed that teachers believe formative assessment is embedded throughout classroom instruction to gauge student learning and used to drive instruction. Responses included "helps me better understand what students know prior to teaching a new concept", "means giving a 1 question exit ticket to see who understands and who is still struggling", "On-going, daily observations" and "Formative assessment gives me feedback on the instruction that has taken place in my classroom."

These responses indicate teachers employ formative assessment prior, during, and at the end of instruction to assess students' understanding. Their responses also show formative assessment is used to improve instruction with examples such as "help to guide me in the instruction", "Reteach immediately for misconceptions", and "any data used to drive instruction." These responses represent the non-linear relationship between gathering formative assessment and teaching. There were however, two responses "End of Unit test" and "Observations, quizzes, tests, assessment activities" that were considered outliers as one refers to a summative assessment and the other response blends summative and formative assessments.

RQ2: What benefit do elementary school teachers report about formative assessment in mathematics?

The survey includes questions where teachers indicated their level of agreement or disagreement. Table 3 shows teachers' perceptions of the benefits of formative assessment.

Table 2
Codes for Participants' Description of What Formative Assessment Means

Code	Frequency in Grades K-2 (18 participants)	Frequency in Grades 3-5 (35 participants)	Total Frequency (53 participants)
Assess learning	8 (44.44%)	27 (77.14%)	35 (66.04%)
Differentiate instruction	10 (55.56%)	9 (25.71%)	19 (35.85%)
Inform or drive instruction	8 (44.44%)	13 (37.14%)	21 (39.62%)
Informal check for understanding	15 (83.33%)	20 (57.14%)	35 (66.04%)
Supports summative assessment (high-stakes tests)	0 (0%)	2 (5.71%)	2 (3.77%)

Table 3
Formative assessment survey statements

Survey Statements	Strongly Agree	Agree	Disagree	Strongly Disagree
Formative assessment in mathematics benefits my teaching.	44	8	1	
Formative assessment in mathematics increases my students' learning.	36	15	2	
Formative assessment in mathematics provides me with more opportunities to increase my students' learning.	42	10		1

Almost all elementary school teacher-participants reported that they found formative assessment to be beneficial and that it increases opportunity and learning for students. One fifth grade teacher responded that they disagree that formative assessment is beneficial and that it increases student learning. The same participant strongly disagreed that it provides more opportunities for the teacher to increase students' learning. One fourth grade teacher disagreed that formative assessment increases students' learning but agreed that it was beneficial and created more opportunities.

The open-ended response revealed more specific details about the benefits perceived by participants. The responses showed that teachers find that formative assessment allows them to be responsive to students' needs in the moment and plan accordingly. Responses included "pinpointing specifically what tools a student needs to master a standard", "teachers can meet students where they are and help them grow", "what to teach next", and "knowing the next steps." Responses like this example "guide to formulate small groups and what gaps students are missing" were consistent, teachers' noted formative assessment helped identify gaps, misunderstandings, and misconceptions. The idea of identifying students' strengths through formative assessment was not stated.

RQ 3: What barriers do elementary school teachers report related to formative assessment in mathematics?

One of the survey questions asked the teachers what disadvantages/barriers are there to formative assessment in mathematics? There were several responses related to lack of time: "Time to analyze/ grade", "Time to create", "Time away from other activities", "Adds to already packed testing", and "Disadvantages - there's already a lot of testing so even though the formative assessments can be quick... It's still another thing to get done and squeeze in." Although teachers perceive formative assessment as beneficial as shown in responses to research question one, there remains concerns about time used in the classroom. In the data there were a few responses such as "There are no disadvantages" and "none" showing consistency between finding formative assessment beneficial and without barriers. There are several responses that provide insight into this difference.

Teachers shared their descriptions of formative assessment and there were differences in their responses that provided more context in the responses for barriers.

- I think the only drawback is analysis of the task. A teacher must ask is this what I just taught or is it a prerequisite skill or is this asking something beyond the standard. You have to be very strategic in picking the right formative assessment.
- Most need to be created to meet the needs of that teacher. I question if it's rigorous to get an accurate measure of what the students can do.
- Creating formatives when they aren't readily available.
- Level of questioning Rigor of questions
- All math work should be seen as a formative assessment that you use to determine student learning.....there are not any barriers.
- I don't think there are any disadvantages to formative assessment. Good teachers are doing this instinctively.

The first three responses show teachers that are considering the efficacy of what they are using for formative assessment, if it is accurately assessing students' knowledge, if it is covering too much content and how rigorous it is. These responses indicate that teachers search for and create material to assess their students. The last two responses show a perception of formative assessment that is less formal and already built into the classroom. The last responses indicate that formatively assessing students is instinctive. These responses provide insight into why a portion of participants find time for creating and analyzing formative assessment to be more of a barrier than others.

A few responses discussed barriers/disadvantages from the vantage point of how formative assessments are used.

- Teachers might dwell on student deficits -teachers might engage students in more low level tasks if specific areas are identified -teachers might spend more time isolating skills and less time helping students seeing connections between concepts -grouping students by perceived ability can be an equity issue -grouping students by perceived ability can lead many students to disassociate themselves from mathematics
- Time - to both effectively implement assessments AND analyze, brainstorm, and plan for instructional activities. Easy to fall into pairing/grouping of students with similar misunderstandings and strengths which limits student potential for growth.

These responses discuss the possible pitfalls of readily using formative assessments. As noted in research question two the idea of formative assessment being used to identify strengths was not mentioned and in the first two responses the idea of becoming overly focused on students' deficits may result in restrictive instruction and groupings that limit growth.

RQ 4: What resources do teachers find useful for conducting formative assessment?

The survey asked teachers to describe their experiences using the following materials to support formative assessment in mathematics: teacher created resources, resources created by a district or school leader, commercially made resources, online resources, digital tools used only to assess students, digital tools used to instruct and assess students. The survey also asked teachers to describe the use and usefulness of the following: commercially made resources (textbook, curriculum), digital instruction on a computer or iPad, teacher-led small groups on current grade content, teacher-led small groups on previous grade content, online resources, and 1 on 1 teaching or tutoring. Table 4 show the breakdown of responses by percentage.

Teacher-led small groups in current content, 1 to 1 teaching or tutoring, and teacher created resources are highly valued and considered useful across the participants. Less than half of the participants view commercially made resources useful. These responses align with participant responses to previous questions that suggest teachers take time to create formative assessments that match their specific needs.

RQ 5: How does formative assessment help teachers differentiate mathematics instruction?

This research question examined participants use of formative assessment data to differentiate their mathematics instruction The previously discussed research questions revealed differences in what teachers consider formative assessment, their benefits, and the barriers. The responses here also showed differences in how the data from formative assessments impacts the learning environment.

Responses were coded as grouped based on level or misconceptions, no grouping, and flexible grouping. The responses were mostly split between grouping by level and flexible grouping with only a few noting that they do not group their students stating “We do not group students using assessment data” or “I typically do not group my students in mathematics. I feel that all students can benefit from the discussion we have at all levels. “The teachers that grouped by level or misconceptions responded with statements such as:

- Students receive instruction in the strategies they are lacking during small group.
- We use group rotations within my group.... each group meets with me.... my lower-level students get a reteach, whereas my higher-level students are taught higher levels of math materials.
- Students are re-taught the lesson, or they are assessed to see what mathematical skill they have not mastered. We try to find what student is missing in his/her math skills so those gaps can be filled.
- Grouping with FA allows students to go above and beyond their learning because they aren't "held back" from the slower learners who need more practice. I can water it down or juice it up depending on the level of knowledge for each group.
- It normally means that those who "get it" - can work solo or in a group on a math group project (still related to what we are doing) - while the others work on something a little.

These responses indicate support for the perception of formative assessment as a tool to find areas of challenge and remediate based on those targeted needs. The responses show students are identified as higher or lower and are grouped accordingly. The last response alludes to a watered-down curriculum based on formative assessments.

There were distinctions made in the responses coded as flexible grouping. Some examples of those responses are:

- My groups are fluid - so if they quickly master the skill, they are moved into a different group. The groups are always changing and the children LOVE it!
- When using data to group students, we may do it in a variety of ways. Sometimes we may group students based on the strategies that they

Table 4

Participants Perceived Usefulness of Resources for Formative Assessment

	Teacher-created resource	Resources created by a district or school leader	Commercially-made resources	Online resources	Digital tools used only to assess students	Digital tools used to instruct and assess students	Commercially made resources (textbook, curriculum)	Digital instruction on a computer or iPad	Teacher-led small groups on current grade content	Teacher-led small groups on previous grade content	Online resources	1 on 1 teaching or tutoring
Very Useful/ Useful	91%	63%	49%	72%	58%	62%	48%	60%	92%	76%	79%	89%
Somewhat Useful	6%	19%	37%	19%	22%	18%	26%	26%	2%	6%	15%	3%
Not Useful	0%	6%	9%	3%	9%	8%	9%	6%	0%	9%	3%	2%
N/A	3%	12%	5%	6%	11%	12%	17%	8%	6%	9%	3%	6%
Total	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

are using. Sometimes we may group students based on a common need. Sometimes we may group students together heterogeneously, so that the thinking of students can nudge the thinking of the whole group or help students to see connections.

- My groups are super flexible, I always work in small group so they see it like something j We typically employ differentiation strategies that allow students to use their strengths, not assume competence and group based on trailing data. A focus on equity means WE need to see all students as capable and provide opportunities that have multiple entry points. We monitor student efficacy and have professional discussions about student identities towards mathematics. Just normal.
- So this differentiation doesn't always result in student grouping of students with like misconceptions or strengths. FAs (formative assessment) allows me to see where some students are struggling, and others are progressing. It pushes me to examine at how I am delivering the content and how it is being received by my students. Sometimes it means reteaching... sometimes it means pairing students with different strengths and misconceptions to help them progress through the problem-solving aspects of these math skills. It's very easy to fall into the trap of "below grade level, on grade level, above grade level" when grouping students by formative assessment data. The challenge lies in using the data to differentiate the approach and instructional activities after the assessment.

In these responses there is attention given to students' misconceptions and providing instruction that supports student growth; however, the responses indicate that the groupings are also based on strengths, use of strategies, equity, differentiation, and change often.

Another survey question asked teachers, what types of instructional activities/resources do you use for differentiated mathematics instruction. Within the responses several digital platforms were identified as a resource for differentiation. The most noted digital resources were IReady, Khan Academy, Moby Max, and Prodigy. Teachers identified country, district and state provided resources such as NC tools for Teachers and NCDPI Tasks as useful for differentiation. The three most noted instructional strategies were the use of math games, small group instruction and manipulatives for reteaching.

Discussion and Implications

This study contributes to the current literature as it provides insight into the use of formative assessment in mathematics for elementary students. There was notable variability in teachers' responses to survey questions. Research suggests formative assessment is a practice that supports mathematics learning for students of all abilities (Gezer et al., 2021; Hattie, 2009; NCTM, 2014); however, it is important to consider how formative assessment is perceived, implemented, and used by teachers. Black and Wiliam (1998) emphasize that using formative assessment correctly would be focused on student learning. With this focus as a lens there would be expected variability in the responses as teachers would be discussing implementation of formative assessment that is centered on their students and their environment. There is variability in responses that remain aligned with research on effective use of formative assessment and some that may be somewhat misaligned.

Participants responded to questions related to how they use formative assessment to differentiate instruction. Small groups were discussed by most participants. Small groups designed to reteach material based on misconceptions revealed in formative assessment data aligns with the goals of using data to support students learning and adapt instruction (Confrey, Toutkoushian, & Shah, 2019; Johnson, Sondergeld, & Walton, 2019; Wilson, 2018). It was the participant responses that conveyed a rigidness toward ability grouping that seems to veer from the recursive relationship of instruction, analysis, and goal setting described by Conderman and Hedin (2012). Flexible groups that change often allow for students to bring different strengths and discourse to their peer interactions. It also prevents students from internalizing negative perceptions of their own ability. Andersson and Palm (2017) added dimensions to Wiliam and Thompson's (2007) framework that emphasize that students are learners and peers. Students should be involved in monitoring their progress and supporting their peers.

Most participants responded that formative assessment was beneficial for student learning, teaching, and providing opportunities for students' learning. When asked to describe what formative assessment means to their teaching and to consider if there are barriers to formative assessment the responses revealed differences in implementation that may contribute to barriers. Bahr and Garcia (2010) describe formative assessments as informal activities that reside within instruction; they provide examples such as exit slips, journals, and discussion. Several participants gave these examples when describing how they implement formative assessment. One of the participants that noted observations, progress monitoring, and not using formal assessments when describing what formative assessment means to their teaching of mathematics also responded none to barriers. It appears that participants that were creating rigorous formative assessments that were more formal were also finding design, implementation, and grading to be a burden.

Over half of the participants highlighted digital instruction as very useful and specifically named IReady, Khan Academy, Moby Max, and Prodigy as platforms that were used for differentiation. Pilli and

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Aksu (2013) suggest that technology tools like these offer immediate feedback. These platforms have the potential to foster formative strategies that have been shown to support student learning such as self-regulated learning (Dignath & Buttner, 2008) and self-assessment (Panadero & Jonsson, 2013). They also may address some of the barriers to implementing formative assessment that were shared by participants.

Formative assessment is focused on student learning, provides ongoing feedback, and provides teachers with insight into student thinking that should guide their instructions. It is a responsive practice rather than standardized, therefore, differences in implementation and use were to be expected. It is important to examine where formative practices deviate from the research-based framework that has shown to improve learning outcomes for all students. Responses to the question of barriers indicate there are areas teachers need support. If barriers to formative assessment are perceived as outweighing the benefit to students, teachers may choose not to engage and grow in the practice. Technology and specific platforms may be an effective part of offering support; however, they must be examined in the same way practices within the classroom are to ensure alignment with formative assessment research.

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Can Music Support Calculation Skills? A Pilot Study Using Electrophysiological Measures*

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Abstract

The purpose of this study is to electrophysiologically assess the effect of an individualized education program supported with musical activities on the success of primary school students with computational difficulties. A mixed-methods design consisting of qualitative and quantitative methods was employed as the research model. The research group consists of four students with mild special learning difficulties and learning difficulties in mathematics, among the students attending the third and fourth grades at a primary school in Kütahya city center. By examining the primary school mathematics curriculum, 12 mental processing gains were identified for addition and subtraction, one of the third-grade mental processing gains. An assessment form, training module, and math songs were prepared in line with these acquisitions. An assessment form was applied to the participants as a pre-test and post-test to determine the effect of the training module and math songs on the students' success.

Additionally, electroencephalogram (EEG) of the participants were recorded before and after the training module, and math songs were applied for 12 weeks. During the EEG recordings, 10 questions were asked to the participants that would enable them to make mental operations. The power densities of the EEG data were calculated using the Welch method in the MATLAB. To analyze the qualitative data of the research, descriptive analysis technique was used. As a result of the study, when the effects of the evaluation form of the participants' mental processing skills were examined, it was seen that the training module and math songs positively affected the mental processing skills of the students. In addition, it has been shown that the prepared training module and math songs increase the success of the participants, supported by electrophysiological evaluations.

Keywords:

Computational Difficulties, Electroencephalography, Mathematics Teaching, Mental Operations, Mathematics Learning Disability

Introduction

While learning can be defined as the acquisition of knowledge, the problems that arise when the individual has difficulties acquiring knowledge can also be expressed as learning difficulties (Korkmazlar, 1999). Learning difficulty is a structural and developmental problem that occurs in reading, written expression, arithmetic and academic, although mental development is generally expected. Geary (2004) argued that although the potentials of individuals measured by intelligence tests are normal or above normal, the failure to achieve the expected achievements in standardized achievement tests and the persistence of this difference in success for two years is an indication that individuals have mathematical difficulties. Mathematics is a challenging course that includes different areas such as arithmetic, solving arithmetic problems, geometry, algebra, probability, statistics, and calculation. This situation requires the use of skills such as quantity perception, symbolic analysis, memory, visuospatial capacity, logic related to various basic skills.

The difficulty experienced in any of these skills or using skills together is expressed as a Mathematics Learning Disability (MLD) (Karagiannakis et al., 2014). These students having MLD and have poor academic performance need to put more effort into their learning processes than their peers (Bintaş, 2007). After students with MLD develop their mathematical skills and adapt to the general education system, methods suitable for their learning characteristics should be adopted to follow the topics in the mathematics curriculum at the same level as their peers (Woodward & Montague, 2002). In addition to receiving their education as full-time inclusive students in regular classes, students with MLD also benefit from support education classroom service in mathematics lessons within the scope of special education (Mutlu, 2016). Individuals with MLD have differences compared to other students and different aspects from individuals with this diagnosis. In line with these differences, an Individualized Education Program (IEP) is prepared for students with MLD.

IEP is a program prepared by a team of experts, including a classroom teacher, for students with special needs, in which different experiences, environments, staff and working hours are required for the individual to display the physical, social, affective, and cognitive behaviors needed for the social norms that are expected to show (Özyürek, 2004). The IEP preparation team is formed according to the school's facilities where the student is located. Although the members legally required to be in the team have been determined, an IEP team is generally formed at schools, including the classroom teacher, guidance teacher, field teachers who teach the other lessons, and the school administrator (Diken, 2014). According to Olson and Platt (2004), IEP is a written record of the services to be provided to students with special needs. Besides, it was concluded that the participants found themselves inadequate in many areas, their self-confidence was built following special education, their attitudes towards mathematics became positive, they observed progress related to the processing skills of students, and supported students in terms of memory skills and retrieval (Temur, 2021).

When all these educational processes are examined in the context of music, it is seen that music might be important for people at every stage of their lives. People live with music from infancy to old age. Almost every child is born with average musical abilities and talents in other academic fields. For this reason, music is an effective and important technique used to support the development of language, motor and cognitive functions as well as improving children's musical abilities (Crncec, Wilson and Prior, 2006). The use of music in mathematics teaching increases academic success. There are many studies showing that music increases mental capacity, and accordingly, the use of music in mathematics teaching increases the academic success of students. Especially in basic mathematics education, the use of music and math songs is very important (Whitehead, 2001).

Yoshida (2005), in his study "The Role of Music in the mathematical performance of high school students with moderate learning disabilities", investigated the effect of listening to music in the background during the math tests of students with learning disabilities, and revealed that music positively affects the mathematical performance. In addition to academic success, music also contributes to students at many points.

De León-Esparza (2019) shows that individuals tend to increase their understanding of the lesson while listening to their favorite music, which reflects a higher level of attention and better focus during lesson practice.

Thanks to music education received, according to some studies the communication of individuals can be healthier, regular, effective, and productive (Uçan, 2005). Also, integrating mathematics with music does not require musical practice or expensive equipment (Edelson & Johnson, 2003). Turan (2006) revealed that musical studies contributed to the social, physical, and mental development of students with special educational needs, but also to the development of students' self-confidence, their self-confidence in other areas where they feel lacking in success and increase their academic success. In addition, music is very effective in developing children's mental capacities and comprehension skills. Thanks to the correct and appropriate use of music in the education of children with MLD and the need for special education, positive effects can occur on these children. Especially the educational processes of children with MLD can be supported by musical stories and songs (MEB, 2014). When the relationship between music and mathematics considered, it is seen that musical activities can offer children different opportunities to support basic mathematical skills. Santos-Luiz and colleagues (2016) revealed long-term associations between music training and academic achievement with music students performing academically better than non-music students. In addition, there might a developmental sequence in the acquisition of basic mathematical skills. At this point, by supporting children's use of a mathematical language with musical activities, a facilitating effect can be created in the acquisition of mathematical thinking and concepts (Dikici, 2002). Moreover and Cheek and Smith (1999) and Wetter and colleagues (2009) found a positive relationship between being engaged in music activities and overall academic achievement. Mehr and colleagues (2013) studied the effectiveness of music training enhancing spatial abilities and mathematics. Thus, studies seem to indicate a relationship between music, intelligence, and learning with music potentially positively contributing to child's brain development. Shaw claimed that music positively affects learning, especially mathematical and some abstract concepts (Shaw, 2003). However not all studies agree on the educational advantage of additional musical activities. Sala and Gobet (2017) revealed that compared to random-effect sizes the impact of music training was very small, with a slightly greater effect size on memory-related outcomes. They concluded that music training did not reliable enhance children's academic skills, making additional studies indicated to understand the nature and origins of the relationship between music and mathematics. According to Sığırtmaç (2005), children can support their ability to match mathematical concepts, using tone of voice, pairing sounds with each other, and matching sounds and instruments one-to-one through musical activities. The names of the children participating in the activities can be used in rhythm studies. In addition, objects can be grouped according to their sounds to develop classification skills in mathematics. While conducting a study on children, a long process may be necessary for children to hear, learn, remember, and repeat to behave according to the prepared model. The use of rhythms and melodies in songs facilitates keeping counting in mind instead of memorizing. Also, during participating in the group learning activities, the students are able to increase their motivation (Hima, 2019). While performing a mental workload, music can have a very important role in affecting the attention and concentration state of the brain (Teixeira, 2018). In this study, it has been concluded that the designed math songs have a facilitating effect on the retention of what has been learned by shortening the process required for the acquisition of mental operations acquisitions. In addition, it has been concluded that the math songs designed in this study have positive effects on mathematics teaching. This result is parallel with the results of the study by An (2013) examining the teachers' methods of integrating music into regular mathematics lessons and the effects of music-mathematics interdisciplinary lessons on primary school students' mathematical abilities in modeling, strategy, and practice. In this study, the education process of the training module, which was prepared by considering the facilitating educational functions of music with students diagnosed with MLD and had calculation difficulties, was supported by mathematical cubes, number base blocks and musical activities that embody mathematical operations. In this context, mathematics songs were prepared by taking expert opinion by acquiring mental addition and subtraction from the primary school third-grade mathematics lesson.

In the context of the relationship between the brain and learning, it is seen that the primary purpose of using brain research in education is to enable educators to comprehend what kind of potential the brain has, what it can do, and which emotions can cause what kind of effects on the brain (Caine & Caine 1990). Interdisciplinary studies aiming to explain how learning takes place in the human brain aimed to understand the nature of learning, examining how one cell connects with another during learning, which parts of the brain are active during this time, and tried to understand how the events in these regions are related to each other (Goswami, 2004). Cognitive neuroscience provides measurements of brain activities using some tools such as Functional Magnetic Resonance Imaging (fMRI) and EEG to interpret brain activities occurring in different states of mind and to understand how cognitive functions are (Dündar, 2014; van Bueren et al., 2021). Studies showed that the human brain constantly produces electric current at very low intensities and spreads the electric current it creates in waves. These bioelectrical potentials, obtained from the brain's neural activities, can be measured using EEG (Tosun, 2004). EEG is expressed as a collection of spontaneously occurring neuroelectric events in regions close to the brain surface (Levy, 1984). EEG is a tool that records the electrical activities in the brain, and electrical effects in the brain can be measured with the help of electrodes connected to different parts of the scalp (Sousa, 2001). These measured values are analyzed and interpreted by experts.

This research aimed to support the IEP, prepared for primary school students with calculating difficulties and musical activities, and reveal the expected difference in these children's mathematical achievement and mental activities.

First, students with computational difficulties among the ones who are diagnosed with mild MLD and have applied to special education and rehabilitation centers are determined voluntarily, and it is aimed to electrophysiologically evaluate the effect of IEP, supported by musical activities, on the success of these students with computational difficulties. The research questions of the study are as follows:

1- What is the effect of the designed IEP on the success of primary school students with computational difficulties?

2-What are the situations encountered during the implementation process of the designed IEP with children having computational difficulties?

3-Is there any difference between the EEG data of the children who received IEP supported by musical activities and the EEG data of the children who received IEP only?

In this sense, it is thought that the results of this study will contribute to the literature and provide information about the mental processing performance of children with MLD. In addition, it is thought that the IEP prepared for these students with MLD will support the planning process and shed light on further studies to be conducted in the field.

In this context, the study results will contribute to the literature and provide information about the mental operations performance of children with MLD, supporting the re-planning of the mathematics learning processes of these students, that different educational programs can be developed for individuals with MLD with the methods used, and that various educational programs can be developed in the field. It is thought that it will shed light on future studies.

Method

This study aimed to electrophysiologically evaluate the effect of IEP supported by musical activities on the achievement of primary school students with computational difficulties. In the study, with the support of IEP, which was prepared with musical activities for primary school students diagnosed with MLD and had calculating difficulties, the change in these students' mathematics achievement and mental activities was revealed.

Research Model

This study aimed to evaluate the performances of the mental addition and subtraction operations of primary school students who were diagnosed with MLD and had computational difficulties in the third-grade mathematics lesson using the training module and math songs in the education process, and EEG recording was performed before and after the education process. Thus, many studies were examined. In this way, due to the multidimensional nature of events and phenomena, mixed design research was chosen as the research model, and qualitative and quantitative approaches were adopted. Mixed-pattern studies used both qualitative and quantitative methods to examine the research problem comprehensively and its many dimensions together (Yıldırım & Şimşek, 2013).

Leech and Onwuegbuzie (2009) argued that mixed research includes collecting quantitative and qualitative data on the same basic phenomena in a single study or more than one study series analyzing this collected data, synthesizing the findings, and then making inferences.

Research Process

In the process of clarifying the research problem, first, a comprehensive literature review on MLD was conducted. Based on this, it is seen that students with MLD also have computational difficulties. One of the measures that can be taken to overcome the computational difficulties experienced by students with MLD is to include musical activities in IEP. After taking expert opinion, an evaluation form prepared by one of the researchers was used to determine students' mental processes with MLD. In developing the evaluation form, first, literature review was conducted. Then, the MEB 1-4 Grades Mathematics Curriculum (2017) was examined, and the achievements for third-grade mental operations were determined. As a result of this, 12 mental process gains were determined, eight of which were mental addition and four of which were subtraction. A multiple-choice draft evaluation form consisting of 24 items was prepared by these achievements. Expert opinion was taken to examine the content validity of this form, clarity in terms of language and expression, and intelligibility.

As the evaluation form were applied to the participants individually, the number of questions in the evaluation form was taken into consideration, and the evaluation form, which was prepared as multiple-choice, was re-prepared during the research process as an evaluation form consisting of open-ended questions by making various arrangements.

In addition to the evaluation form, a training module and math songs related to each achievement were prepared in accordance with the mental processing achievements, which took 12 weeks to be applied to each student. While the training module and math songs were being prepared, third-grade mathematics textbooks were examined, and activities were designed in accordance with the achievements of the course. The prepared training module and math songs were sent to the experts to obtain their opinion.

In line with the opinions and recommendations of the experts, the training module and the songs were reexamined, and necessary arrangements were made. In designing math songs according to the third-grade acquisitions, it was prioritized that they should be suitable for mental processing gains and have fun content ideal for children. After completing the draft work of the math songs, the field expert was also consulted. After receiving feedback from the field expert, necessary arrangements were made, and the math songs took their final form.

Alpha, Beta, Theta, Delta, and total powers were calculated by performing Welch analysis of the participants' EEG data though MATLAB to evaluate electrophysiologically what effect the students' achievement and brain waves had when IEP supported by musical activities was applied to primary school students with computational difficulties in their mathematics learning processes.

Study Group

The study group consists of students diagnosed with mild level (20%) MLD and have learning difficulties in mathematics lessons among the students attending the third and fourth grades in primary schools located in Kütahya city center. The students in the study group were determined by the purposive sampling method. Students who attend special education and rehabilitation centers in Kütahya and have computational difficulties were determined. Four students were selected among these determined students on a voluntary basis.

Data Collection Tools

In the first semester of the 2018–2019 academic year, the evaluation form and training module were applied to four volunteer students aged 9–11 who were attending the Special Education and Rehabilitation Centers of the Ministry of National Education in Kütahya and were in compliance with the research criteria.

After the evaluation form, training module and math songs were prepared, the volunteer participants were determined according to specific criteria. During the determination, factors such as students' volunteering, mild (20%) MLD and difficulty in the calculation were effective. Before performing the application, one of the researchers informed the participants and their parents about the research topic and research process. Before starting the application, one of the researchers applied the evaluation form as a pre-test to four primary school students aged 9–11 with MLD and recorded the students' correct, incorrect, and blank answers.

Data on learning difficulties experienced by each student were obtained from special education teachers working in special education and rehabilitation centers. Then, interviews were conducted with the families of these students with MLD by setting appropriate meeting times. In the interviews with the parents, information about the study was given.

Before the research, although 10 parents were interviewed, five parents did not want their children to participate in this study, considering that they even came to the Special Education and Rehabilitation Center reluctantly. One of the five students who wanted to participate in the study voluntarily was diagnosed with epilepsy and, as a result of the data obtained during the study's EEG recording, was excluded. As a result of the interviews with the parents of the remaining four students, the necessary legal permissions were obtained from the students and their parents who wanted to take part in the study voluntarily.

An electrophysiological evaluation was made by performing an EEG recording before and after the training to see the effect of the training module, supported by musical activities given for 12 weeks, on the students' brain waves. In this evaluation, EEG data were analyzed by performing Welch analysis in the MATLAB. During these EEG recordings, 10 questions were asked to the participants to enable them to make mental operations. From the moment participant saw the question, EEG data were recorded when participant made a solution in participant's mind. Therefore, from the moment participant saw the question and started to think in participant's mind, the waves formed in the brain were recorded in the computer environment through the EEG device.

Before the EEG recordings, a sample recording was performed with the personnel in the EEG laboratory. Thanks to these sample recordings, the most suitable conditions for the research were provided by taking the neurologist's opinion. In this study, a unique database was obtained by recording the EEG data with a Nihon Kohden 1200 digital EEG device that can capture 16 channels with high quality and reliability.

Following the mental processing gains, 10 questions determined with the expert were asked by one of the researcher by showing question cards to the participants during the EEG recording, giving one minute for each question. After each one-minute question, a 30-second break was given, and then new questions were started. Because of this situation, the first 30 seconds of the recorded EEG data were taken from the moment the questions were asked.

Thus, five numerical values were obtained as Alpha, Beta, Theta, Delta, and total power from each question asked for each EEG recording of each student. Thus, a total of 100 data, including the first and last recordings, were obtained, five each consisting of 10 questions. As a result, 400 power data is obtained from four participants. Then, the data recorded with the digital EEG recording device were transferred to the computer environment and converted into forms on which appropriate analyses could be made.

Analysis of Data

Analysis of quantitative data

Various analysis methods have been developed for the classification of EEG signals. One of the most widely used of these analysis methods is the Welch analysis method, in which the power spectrum density is calculated using non-parametric methods (Faust, 2008). In the EEG analysis method, mathematical tools are used to analyze the data. The characteristics of the EEG signals to be analyzed can be found by the power spectrum density (Subasi et al., 2005). The power densities of the frequencies between 1 and 48 Hz of the EEG data were calculated using the Welch analysis method. Neurofax EEG System was used for EEG recording. In this study, it was deemed appropriate to conduct Welch analysis for the research data by the relevant field experts. Fast Fourier transform (FFT) algorithms based on Fourier transform are generally applied to these analyses. At this point, the Welch analysis is accepted as one of the well-known non-parametric power spectral density estimation analyses (Tosun, 2018). While analyzing the EEG data, MATLAB (MathWorks, USA) program was used. MATLAB is a high-performance software primarily written for technical and scientific calculations, including numerical computation, graphical data representation, and programming. General usage areas of the MATLAB program can be summarized as Mathematics and computational processes, algorithm development, modeling, data analysis, scientific and engineering graphics, and application development. Evaluating EEG data is a complicated task. The data obtained from EEG shots can be affected by several physiological conditions such as hunger, age, wakefulness-sleep state, and mental state. To get more reliable results in evaluating EEG data; during the EEG recording, the conditions required for the recording were tried to be provided in the best way, and the physiological conditions of the participants were also taken into consideration.

Analysis of qualitative data

To obtain information about each student in the study group, a literature review was conducted by one of the researcher. In the light of the information obtained, the one of the researchers made various observations before starting the application to know the environment he would practice. In this context, the observation technique was used to describe the behaviors occurring in any environment or situation in detail. The observation method can present one of the researchers with a detailed, comprehensive, and more extensive picture of behavior that occurs in any environment (Yıldırım & Şimşek, 2013). The researchers met the students and their families participating in the research before the study and talked with the participants before starting the practice and got used to the researchers. The researchers also chatted with the students before and after each application and had the opportunity to learn about the students' thoughts about the mathematics lessons and learning. During the observation, the researchers did not use any form. During the research process, 12 lesson hours with each student were recorded with a camera to record the students' performances during the application. First, the camera recordings were watched together with the researchers and experts. Then, the camera recordings were transcribed by the researchers. In the data analysis, the names of the research participants were not used. Instead, they were expressed as Participant 1, Participant 2, Participant 3, Participant 4. In this sense, the researchers met the students and their families who participated in the research and talked with the participants before starting the application and made them get used to the researchers. The researchers also chatted with the students before and after each application and had the opportunity to learn about the students' opinions about the mathematics lesson and learning.

Descriptive analysis techniques were used to describe and summarize the data obtained as a result of the researcher's observation in the training module and the application of math songs. The research data obtained in the descriptive analysis were first described systematically and clearly. These descriptions were then explained and interpreted by the researchers. The resulting cause-effect relationships were also examined, and some results were showed (Yıldırım & Şimşek, 2013). First, frequency tables were prepared and analyzed in the analysis of the data obtained with the evaluation form. With these analyzed data, it is aimed to reveal what kind of change there is in the pre-test and post-test evaluations of the participants. In the pre-test and post-test, the participants' answers were specified as True or False, and individual assessments of each student were made.

Validity and Reliability of the Research

A literature review was conducted to ensure the validity of the questions in the Validity and Reliability Evaluation form. It was then prepared in accordance with the purpose of the research by taking the opinions of the relevant experts. The questions in the evaluation form were applied after the thesis advisor, faculty member, and field experts were determined that they could fully serve the purpose of the research. During the research process, additional measures were taken to ensure the validity and reliability of the research findings. The researchers obtained permission from the National Education Directorate and the participants' families to conduct this study. Not to interrupt the research, the home environment, which is the environment where the participants can feel most comfortable, was determined as the practice environment. Camera recording was made to avoid data loss in the research, and the researchers converted the data into text. The study was approved by Kütahya Health Sciences University Faculty of Medicine Clinical and Laboratory Research Ethics Committee with the number 81469268-900 dated 13.06.2019. Before the EEG examinations, the participants in the study were informed about the research. Informed Voluntary Parent Consent Form, Parent Statement and Child Consent Form were signed by the participants' families.

Study Results

In the qualitatively designed part, the data obtained from the observation technique using video recordings were presented with the descriptive analysis method. In that part again, expert evaluation was made in the analysis of the EEG data taken while calculating the mind. It was presented within the framework of Welch analysis, one of the signal analysis methods, through the MATLAB program.

Findings Regarding the Effect of the Designed IEP on The Success of Primary School Students Who have Computational Difficulties.

An evaluation form was used to determine the effect of the designed IEP on the success of primary school students with computational difficulties. When the evaluation form findings were examined, Participant 1 correctly answered 18 of the 24 questions asked in the pre-test, which was the first application of the evaluation form. In the post-test, which was the second application of the evaluation form, it was observed that Participant 1 gave the correct answer to 20 questions out of 24, and as a result, the number of correct answers by Participant 1 in the evaluation form increased. Participant 2 correctly answered only three of the 24 questions asked in the pre-test, which was the first application of the evaluation form. In the post-test, which was the second application of the evaluation form, it was observed that Participant 2 gave the correct answer to 21 questions out of 24, and as a result, it can be stated that the number of correct answers by Participant 2 in the evaluation form increased. Participant 3 correctly answered nine of the 24 questions asked in the pre-test, which was the first application of the evaluation form. In the post-test, which was the second application of the evaluation form, it can be stated that Participant 3 gave correct answers to 21 questions out of 24, and as a result, the number of correct answers by Participant 3 in the evaluation form increased. Participant 4 correctly answered 14 of the 24 questions asked in the pre-test, which was the first application of the evaluation form. In the post-test, which was the second application of the evaluation form, it can be stated that Participant 4 gave correct answers to 21 questions out of 24, and as a result, the number of correct answers by Participant 4 in the evaluation form increased.

In the pre-test, which is the first application of the evaluation form, Participant 1 answered three questions correctly, Participant 2 answered nine, and Participant 3 answered fourteen questions correctly. In the post-test, which was the second application of the evaluation form, these participants gave correct answers to 21 questions out of 24. As a result, it is seen that the increase in the number of correct answers of all participants is significant.

The first 18 questions of the evaluation form consist of questions about the mental collection process, and the remaining six questions are prepared for the mind extraction process. When the prepared questions are examined, it is seen that the participants mostly have difficulties in mental operations, which are shown with number models and written side by side. In the post-test, all participants correctly answered the fifth and seventh evaluation questions, which were prepared using the representations of numbers with models. In addition, it was observed that the participants were wrong in the questions in which one of the totals and the result were given and the totals were not given. Likewise, the participants had several difficulties in the mind subtraction process, which asked the participants to find the subtraction and remainder by providing the remainder. In the light of the data obtained from the evaluation form after the 12-week training, it is clearly seen that there is a significant increase in the correct number of participants.

It was observed that the participants had more difficulties in mind subtraction than in mind addition processes. After the training given, there was a significant increase in the number of truths obtained from mind subtraction. As a result of applying the evaluation form as a pre-test, a total of 44 correct answers were obtained from four participants. In comparison, a total of 83 correct responses were obtained from the participants as a result of the application of the evaluation form as a post-test.

Findings Regarding the Situations Encountered During the Implementation Process of the Designed IEP with Children with Computational Difficulties

The situations related to each outcome were interpreted separately to determine the situations encountered during the designed IEP with children with

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computational difficulties. Regarding the acquisition of mentally adding up two natural numbers, the sum of which does not exceed 100, the participants forgot to calculate the hand while doing the addition. In addition, the participants had difficulties while doing the addition operations given side by side. When they wrote the given numbers one under the other and added them together, they achieved the correct results by easily doing the operation.

Regarding the acquisition of mentally adding a three-digit number and a one-digit number, the participants made finger calculations while adding. As they did not know the representation of the numbers with the models by dividing them into units, tens and hundreds, they could not do the addition process correctly with the parts of the numbers indicated by the models. However, after learning the representations of numbers with models, they could correctly perform addition operations. Regarding the acquisition of mentally summing a two-digit number that is a multiple of 10 and a three-digit number that is a multiple of 100, the participants counted tens by 10 and counted one 10 less or more.

Regarding the acquisition of "Makes mental addition using the rounding strategy," the participants had difficulties rounding the numbers to the correct tens because they did not know how the numbers were rounded to the nearest tens.

Regarding the acquisition of "Makes mental addition by using the number pairs strategy," the participants made finger calculations while doing the addition with number pairs. It is seen that the participants confuse the place values when adding about the acquisition of "Makes mental addition by using the place values strategy." Regarding the acquisition of "Makes mental addition by using the adding on strategy," the participants could do the addition operations more easily and accurately as they could see the numbers concretely on the number bar in the addition processes given by the length model. In addition, the participants marked the given additions one by one using a pencil on the number bars and counted all the numbers without getting bored. Regarding the acquisition of "Makes mental addition by using the segmentation strategy," the participants had difficulty in separating the two-digit numbers using the segmentation strategy.

The participants did not know how to subtract through number models regarding the acquisition of "Makes two-digit numbers that are a multiple of 10 from two-digit numbers by mind subtraction." When subtracting two-digit numbers that are a multiple of 10 from two-digit numbers, they learned that the remainder would result from subtraction by placing a cross on the column representing each ten. Then, the participants could do the subtraction more easily and accurately when the subtraction operations were expressed with numerical models, and the given subtraction was concretized. Regarding the acquisition "Makes mental subtracting natural numbers multiple of 10 from three-digit natural numbers multiple of 100," the participants tried to perform the subtraction given as an addition operation, forgetting that they were doing subtraction while performing subtraction on natural numbers.

Regarding the acquisition of "Makes subtraction by using the strategy of adding on," the participants could not correctly remember which of the numbers given in the subtraction process is the subtractive number, which is the subtracted number, and which is the remaining number. In the subtraction operations, the researchers reminded the participants that the missing number can be found by adding the number and the remaining number in the questions asked to see the decreasing number by giving the number and remainder. Regarding the acquisition "Makes mental subtracting operations using the strategy of breaking numbers into parts," the participants made various mistakes while performing mental subtraction by using the strategy of breaking numbers into parts.

In addition, throughout the training process, Participant 1 was a very willing student in both the training module part and the singing math songs part of the research. Participant 2 was an introverted student who did not like to talk much, had a somewhat shy nature, but willingly participated in the training module and materials. Participant 3 was a shy student who was somewhat reluctant to do a training module, wants the activities to end quickly, but was eager to sing and learn math songs. Participant 4 could be defined as a student who both did not want to sing the math songs and participated in the training module activities reluctantly.

Findings in Terms of Determining the Difference Between the EEG Data of Children Who Received IEP Supported by Musical Activities and the EEG Data of Children Who Only Received IEP

The EEG data obtained from the participants were analyzed and interpreted to determine whether there was a difference between the EEG data of the children who received IEP supported by musical activities and the EEG data of the children who only received IEP. Welch analyses of the power data obtained from the participants during the EEG recordings were made in the MATLAB, and the data revealing the results were examined. As there is excessive Alpha activity in children with MLD, decreases in Alpha power are interpreted as an increase in mental activities in the brain (Chabot, 1996).

There was a change in Alpha, Beta, Delta, Theta and Total Powers obtained from the questions asked during the EEG recording of the participants. In Participant

1, there was a decrease in the values obtained from eight questions in Alpha power, and an increase in Beta power in four questions. In addition, delta strengths decreased in six questions. There was a decrease in theta power at five questions. Also, there was a decrease in total power in only two questions. In Participant 2, the values obtained from eight questions decreased in Alpha power. There was an increase in beta power again in two questions. Delta power decreased in seven questions. There was a decrease in theta power in four questions. There was a decrease in total power in seven questions. In Participant 3, the values obtained from only two questions decreased in Alpha power. However, a total of seven questions increased in Beta power. Delta power decreased in four questions. There was a decrease in theta power in the four questions. There was a decrease in total power in seven questions. In Participant 4, the values obtained from only two questions decreased in Alpha, Beta, Delta, Theta and Alpha power. Beta power increased by nine questions. Delta power decreased in one question. There was a decrease in theta power in one question. There was a decrease in total power in the three questions.

Alpha rhythm is seen in awake, ordinary, and calm people. During the sleep, the Alpha rhythm disappears. If the awake person directs his attention to something special, such as a mental activity, a higher frequency Beta rhythm occurs instead of Alpha waves (Yazgan & Korürek, 1996). As there is an excessive Alpha activity in children with MLD, decreases in Alpha power are significant as an increase in mental activities in the brain. Beta wave is the brain wave observed while awake. It is obtained mainly from the anterior parts of the brain (Başar, 2012). It also occurs when the human brain is exceptionally dense. When a person is exposed to too many external stimuli, an increase in beta waves occurs in the brain. Beta wave is active when eyes are open while listening, thinking, solving analytical problems, making decisions, making judgments, and processing the information around (Aydemir & Kayıkçıoğlu, 2009).

Alpha wave is important in learning and using information activities and it is seen that it decreases while the individual is performing thinking and problem solving processes. The beta wave seems to get stronger during analytical problem solving, judgment, decision making and audio listening (Ildız, 2007).

In this study, as excessive alpha activity is observed in children with MLD, decreases in alpha power and increases in beta power are significant as an increase in mental activities in the brain. If awake people direct their attention to something special, such as a mental activity, higher frequency Beta waves are formed instead of Alpha waves. The findings of the study revealed that when the participants are exposed to too many external stimuli, a decrease in Alpha waves and an increase in beta waves occur in the brain.

Conclusion and Discussion

Conclusion and Discussion Regarding the Evaluation Form

After the literature review of the evaluation form was made, the MEB first-fourth grades mathematics curriculum (2017) was examined, and the achievements for the third-grade mathematics course mental operations were determined. As a result of this examination, 12 mental process gains were determined, eight of which are mental addition and four of which are subtraction. A 24-item evaluation form consisting of open-ended questions suitable for these acquisitions was prepared. The following results were obtained from this evaluation form.

Considering that the first 18 questions of the evaluation form consisted of questions related to the mental addition process and the remaining six questions were prepared for the mind subtraction process, it was observed that the participants had difficulties in mental operations, which were shown with numerical models and written side by side, among the most prepared questions. However, after the training, all participants were able to answer the fifth and seventh evaluation questions correctly, which was prepared using the representations of the numbers with the models in the post-test.

It was observed that the participants were wrong in the questions in which one of the totals and the result were given, and the totals were not given. Similarly, the participants had several difficulties in the mind subtraction process, which asked the subtraction and remainder to be found. It was concluded that there was a significant increase in the correct number of participants when considering the data obtained from the evaluation form after 12 weeks of training. Participants had more difficulties in mind subtraction than in mind picking. However, it was also concluded that after the training, there was an increase in the number of truths obtained from mind subtraction. While 44 correct answers were obtained from four participants because of the application of the evaluation form as a pre-test, a total of 83 correct answers were obtained from the participants as a result of the application of the evaluation form as a post-test. The difference between the pre-and post-training was determined through the evaluation form. In a study by Wisniewski and Smith (2002), the effectiveness of Touch-Math, a mathematics set that aims to teach mathematics to students more easily, in increasing the mathematics achievement of primary school third and fourth-grade students with special

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needs were investigated. The students participating in the research taught mathematics for 14 weeks in the resource room for 45 minutes, 20 minutes of which were on the Touch-Math set. As a result of this research, a significant difference was observed in the duration, and correct answers of the tests applied to the students.

Conclusion and Discussion Regarding Discussion the Training Module

Some results were obtained in the designed training module, and the math songs were applied together. The findings of the evaluation form of mental processing skills of students with MLD revealed that the training module and math songs positively affect students' mental processing skills. In the study, it was observed that the participants' mental addition and subtraction performances increased. The study findings revealed that the mistakes made by the students in writing numbers and the problems they experience while performing mental operations have decreased significantly compared to the initial level. It is thought that this situation results from the prepared training module and math songs.

Another type of mistake that students with MLD and computational difficulties make the most is changing a transaction. At this point, the use of musical activities in mathematics teaching had a positive effect on learning operations from the mind. In addition, it was concluded that the math songs designed in this study had positive effects on mathematics teaching. In the study by Raghubar et al. (2009) the arithmetic performances of primary school third and fourth-grade students with and without MLD were compared. The study examined students' performances in multi-digit addition and subtraction. As a result, they found that students with MLD make more mistakes such as finding close values, moving the hand to another column, not carrying the hand, and not breaking a decimal point (Raghubar, 2009).

Another common mistake that students with MLD and computational difficulties make the most is changing a transaction. In addition, the use of musical activities in mathematics teaching had a positive effect on mental learning processes. In the study about the effect of the use of songs in elementary mathematics teaching, it was found that mathematics teaching supported by music activities using songs in primary school third-grade mathematics lessons resulted in a difference in students' attitudes toward music and mathematics, achievement scores, multiple intelligence areas and memory levels (Bütüner, 2010). In addition, it was reported that the use of songs in mathematics teaching had a positive effect on students' thoughts and opinions about the mathematics lessons. Shaw, Graziano, and Peterson (1999), "Piano and Computer Training Boost Student Math achievement." In their study, it was revealed that the mental abilities of children who received music education in early childhood developed. The contribution of music to the development of mental abilities is stated. Whitehead (2001), in his study "The Effect Of Music-Intensive Intervention On Mathematics Scores Of Middle And High School Students", obtained results indicating that the use of music in mathematics teaching increases academic achievement. He states that there are many studies showing that music increases mental capacity, and accordingly, the use of music in mathematics teaching increases the academic success of students. Especially in basic mathematics education, the use of music and math songs is very important.

Whitehead (2001) obtained results indicating that the use of music in mathematics teaching increases academic achievement. He states that there are many studies showing that music increases mental capacity. Accordingly, it is stated that the use of music in mathematics teaching increases the academic achievement of students. It is known that music contributes positively to the academic success of students of many levels.

Yoshida (2005), in his study "The Role of Music in the mathematical performance of high school students with moderate learning disabilities", investigated the effect of listening to music in the background during the math tests of students with learning disabilities, and revealed that music positively affects students' mathematical performance. In addition to academic success, music also contributes to students at many points. De León-Esparza (2019) shows that individuals tend to increase their understanding of the lesson while listening to their favorite music, which reflects a higher level of attention and better focus during lesson practice.

According to Sığırtmaç (2005), children can support their skills of matching mathematical concepts, using tone of voice, matching sounds with each other, matching sounds and instruments one-to-one with musical activities. In these rhythm works, the names of the children participating in the activities can be used. In addition, objects can be grouped according to their sounds to develop classification skills in mathematics. While conducting a study on children, a long process may be necessary for children to hear, learn, remember, and repeat to behave according to the prepared model. The use of rhythms and melodies in songs facilitates keeping counting in mind instead of memorizing.

While performing a mental workload, music can have a very important role in affecting the attention and concentration state of the brain (Teixeira, 2018). In this study, it has been concluded that the designed math songs have a facilitating effect on the retention of what has been learned by shortening the process required for the acquisition of mental operations acquisitions. In addition, it has been concluded that the math songs designed in this study have positive effects on mathematics teaching. This result is parallel with the results of the study by An (2013) examining the teachers' methods of integrating music into regular mathematics lessons and the effects of music-mathematics interdisciplinary lessons on primary school students' mathematical abilities in modeling, strategy, and practice.

Conclusion and Discussion Regarding EEG Recordings

The findings of the study revealed that the Alpha power of the participants decreased, and the Beta power increased at certain rates. As children with MLD have excessive Alpha activity, decrease in Alpha power are as important as an increase in mental activities in the brain (Chabot, 1996). Alpha rhythm is seen in awake, ordinary, and calm people. In the sleep state, the Alpha rhythm disappears. If the awake person directs his attention to something special, such as a mental activity, a higher frequency beta rhythm occurs instead of Alpha waves (Yazgan & Korürek, 1996). Alpha waves are thought to indicate a relaxed state of awareness that does not require concentration and attention. It is also the most common rhythm in the brain. Most people have an Alpha wave when their eyes are closed. It oscillates and the oscillation decreases as the eyes open for a different sound, excitement, or attention. Alpha waves are usually accompanied by beta and theta waves (Niedermeyer, 1999). The beta wave is usually released in situations such as active thinking and attention, problem solving or focusing on something. It can be blocked by motor activity or tactile stimuli (Sterman et al., 1974).

A decrease in the Alpha powers of the participants and an increase in the Beta powers were also observed at certain rates as a result of this study. A change was observed in Alpha, Beta, Delta, Theta and Total Powers obtained from the questions asked to Participant 1 during the EEG recording. Alpha power obtained from 10 questions directed to Participant 1 decreased in eight questions, while Beta power increased in four questions. Also, a decrease was observed in total power in two questions only. It is thought that the decrease in Alpha power, which is desired by Participant 1 to be willing and interested in the training module and math songs, is effective. Alpha power obtained from 10 questions directed to Participant 2 decreased in eight questions. There was an increase in beta power again in two questions. In addition, it is thought that the decrease in Alpha power, which is desired to exhibit high participation in the activities performed very willingly in the training module activities prepared by Participant 2, is thought to be effective. Alpha power obtained from 10 questions directed to Participant 3 decreased in only two questions, while Beta power increased in seven questions. Participant 3, on the other hand, was bored and unwillingly participated in the training module despite all the efforts of the researchers in the training module activities. However, Participant 3, who did not participate in the training module, willingly participated in math songs. It is thought that these situations effectively decrease the desired Alpha power in only two questions and the increase in Beta power in seven questions when considering the participants' individual differences. Alpha power obtained from 10 questions directed to Participant 4 decreased in only two questions, while Beta power increased in nine questions. Despite all the efforts of the researchers, Participant 4 got bored in the training module activities prepared and participated in the training module without much enthusiasm. It is thought that these individual differences are effective in the decrease in the desired Alpha power and the increase in the Beta power in a certain number when considering the individual differences of the participants.

Aker and Akar (2014) examined the effects of Turkish music makams through the analysis of EEG waveforms. For the application, 15 healthy individuals listened to Turkish music makams. The collected EEG data were separated into subbands by the discrete wavelet transform method. The power densities of each band were calculated using the power spectral density method. As a result of their analysis, they saw the effect of the authorities on the EEG signals in the beta band. At this point, it is seen that studies can be done in many areas with EEG measurements. By making EEG measurements in different fields such as education, medicine, engineering, marketing and advertising, meaningful results can be obtained from the activities in the brains of people, and suggestions and changes can be made about the systems from these results.

For primary school students with MLD to acquire skills for mental operations, first, activities that use various mathematical materials and embody the teaching might be indicated instead of only teaching math based on paper and pencil exercises. The concepts of numbers and operations, which form the basis of mental operations, are actually abstract concepts. For the children to understand these abstract concepts, teaching should be supported with concrete objects and tools. For children with MLD, training on mental operations should first start with activities related to addition. If they succeed in addition-related activities, subtraction-related activities should be started. In accordance with the individual educational needs of students with MLD, the educational process should be supported with mathematical songs when necessary. In addition, in the context of the findings obtained in the study, it is recommended to do it through materials that provide modeling in the teaching of mental processing skills.

This study focused on revealing the difference expected to occur in these children's mathematical achievement and mental activities by supporting the IEP prepared for primary school students with computational difficulties with musical activities. Also, future research can be conducted to examine the different acquisitions at different grade levels at the primary school level. Based on this research, attempts can be made to educate children with MLD and solve the educational process problems. This study has some limitations. The main limitation of the study is that the number of students participating in the study is limited because it includes qualitative research, and the participants were from the same province. Future studies can enrich the existing findings employing qualitative and quantitative methods involving other provinces and involving more participants. In addition, based on this limited number of participants and because of the meta-analysis of Sala and Gobet (2017), we have to be very careful with the claims we make about music enhancing calculation skills. Future studies with more participants and different disciplines can be brought together to contribute to the field to study if music related activities can improve calculation skills in participants with MLD.

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How a Measuring Perspective Influences Pre-service Teachers' Reasoning about Fractions with Discrete and Continuous Models

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Abstract

How teachers interpret and express fractions critically influences their teaching and their students' fraction knowledge. Internationally, the mathematics education community has been studying ways to enhance pre-service elementary teachers' rational number knowledge, particularly fractions. To address the challenge of augmenting pre-service teachers' fraction knowledge warrants theoretical and empirical revisions to standardized practices for teaching fractions. This study investigates how reexamining fractions from a distinctive measuring perspective influences pre-service teachers' reasoning about fractions. For four 75-minute sessions, 46 pre-service teachers enrolled in a teacher preparation program at a university in the United States revisited fractions from a measuring perspective. They engaged in tasks that focused on comparing continuous quantities and identifying relative magnitudes. The data for this study comprise their pre- and post-tests that assessed how they identify and represent fractions with discrete and continuous models. For each model, we analyzed participants' reasoning by attending to their written strategies. Findings revealed three main strategies: partition, construction, and symbolic manipulation. In general, participants expressed more strategies on the post-test for all fraction models. Partitioning was the most frequent strategy on the pre- and post-tests. However, the frequencies of strategies changed after the intervention. For example, with all models, there was an increase in partitioning strategy and a decrease in symbolic manipulation strategy. The results highlight affordances of a measuring perspective to support participants to shift from procedural strategies such as symbolic manipulation to more conceptual strategies to identify and represent fractions.

Keywords:

Fraction Models, Pre-service Teacher Knowledge, Mathematical Reasoning, Measuring Perspective

Introduction

For mathematics educators, how to support students' meaningful learning of fractions has been a significant challenge. Starting in students' initial schooling years, teachers struggle to help them conceptualize and operate on fractions (Zhou et al., 2006). Those who experience difficulties ordering and operating on fractions (Maher & Yankelewitz, 2017), underachieve mathematically, and are unsuccessful in learning higher subjects such as algebra (Fuchs et al., 2017; Siegler et al., 2012; Torbeyns et al., 2015). Evidence indicates that teachers' knowledge predicts students' achievement gains (Charalambous et al., 2020), and, therefore, teachers' understanding of fractions is crucial for students to learn rational numbers and operations on them. However, as Torbeyns et al. (2015) indicate, "systematic studies in Europe and North America point to deficits in (prospective) teachers' content and pedagogical content knowledge of mathematics in general and rational numbers in particular" (p. 7). Recent studies reveal challenges that pre-service teachers (PSTs) have with conceptual and procedural fraction knowledge (e.g., Bobos & Sierpinska, 2017; Busi et al., 2015; Depaepe et al., 2015; Harvey, 2012; Tobias, 2013; Toluk-Uçar, 2009; Utley & Reeder, 2011; Van Steenbrugge et al., 2014), particularly with fraction multiplication and division (Lo & Luo, 2012; Morano & Riccomini, 2019; Olanoff et al., 2014; Siegler & Lortie-Forgues, 2015; Young & Zientek, 2011).

Researchers have based their investigations into the learning of fractions and their operations on two ontological perspectives: partitioning and measuring (Powell, 2019a). The first perspective views a fraction as a relation between parts of a single whole or quantity subdivided into equal portions. This perspective emphasizes counting, leading to the commonly accepted part/whole conception of fractions (Schmittau, 2004). The Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and some mathematicians interested in pre-university and teacher mathematics education (Wu, 2014) suggest this conception for introducing fractions in elementary schools.

Nevertheless, research suggests that this partitioning perspective seriously limits the robustness of students' understanding of fractions (Kerslake, 1986). It encourages the erroneous idea that a fraction only represents two different discrete quantities and hinders understanding improper fractions (Tzur, 1999). While research on fraction knowledge has yielded information about fraction learning's cognitive issues (Behr et al., 1997; Dienes, 1967; Kieren, 1980; Lamon, 2007, 2012; Mack, 1990; Tzur, 1999), a critical limitation has been a focus on initial fraction learning from the partitioning perspective.

Few studies have investigated a known alternative source for fraction knowledge, a measuring perspective. Rather than basing fraction knowledge on counting discrete portions of a single quantity or collection, the measuring conception views fractions relationally. A fraction is a number or ordered pair of numbers that indicates a relation between two commeasurable quantities of the same kind (Davydov & Tsvetkovich, 1991; Gattegno, 1960/2009, 1974/2010). The relation is a multiplicative comparison, where one quantity measures a multiple of the other. In this perspective, learners conceptualize fractions through multiplicative reasoning (Vergnaud, 1983, 1988).

In both the partitioning and measuring perspectives, to help students conceptualize fractions, teachers use manipulatives and representational materials such as folded paper strips, area models, and collections of objects. Yet, few manipulatives allow for a tangible, flexible, and complete model of any fraction and the arithmetic operations on fractions. For instance, some researchers (Lee & Lee, 2019) note how circular models are inconvenient to illustrate fractions with large denominators and do not recommend set models for fraction comparisons. Contrastingly, length models can easily represent fractions of any denominator, fraction magnitudes (i.e., the numerical values fraction symbols represent such as ¾), and fraction operations (Carraher, 1993; Fazio & Siegler, 2011).

With or without manipulatives, learners use symbolic representations of fractions to resolve tasks. The strategies they use have been the focus of research. For example, in magnitude comparison tasks, researchers (Mack, 1990; Erol, 2021) found that students based their comparative judgments on the amount needed to reach one, so to compare 5/6 and 7/8, students used 1/6 and 1/8. Mack (1990) discovered that students judged 1/8 to be larger than 1/6 since eight is greater than six, thus misusing a whole number property for comparing fractions. Incorrectly applying properties of whole numbers on fractions is a common strategy known as whole number bias (Ni & Zhou, 2005). Another example concerns strategies related to locating fractions on a number. Siegler et al. (2011) found that middle school students used numerical transformation or segmentation strategies to locate fractions on a number line. With numerical transformation strategies, students transformed a fraction to an easier one, while segmentation strategies involve partitioning the number line into a certain number of segments. When comparing fractions, other researchers found that procedural manipulations are usually associated with incorrect comparisons and nonrelational thinking. In contrast, strategies based on number sense such as benchmarks and estimation support mathematical reasoning and evidence conceptual understanding (Sengul, 2013; Yang et al., 2009). Therefore, teaching fractions should consider approaches and practices that help learners develop conceptual strategies for representing and operating on fractions.

Our study investigates how a distinctive measuring perspective influences PSTs' reasoning about fractions represented in multiple models. Specifically, we ask this question: How does revisiting fraction knowledge using a measuring perspective influence PSTs' reasoning about fractions represented in rectangular, circular, and set models? To examine PSTs' reasoning about fractions, we attend to their strategies to solve fraction tasks within each of the three models.

In what follows, we present pertinent literature and our theoretical framework, methods, and findings. Finally, we discuss the strategies that PSTs used on pre- and post-tests in light of the current literature and suggest areas for further research.

Pertinent Literature and Theoretical Framework

Current influential perspectives on fraction knowledge have a common origin. Kieren (1980) introduces and analyzes a taxonomy of interrelated interpretations of rational numbers. Concerning fractions, researchers (Behr et al., 1993; Charalambous & Pitta-Pantazi, 2007; Kieren, 1993; Lamon, 2007) widely recognize five standard interpretations: part of a whole, quotient, operator, ratio, and measure. To illustrate, 3/4 as a part of a whole (an area or a collection) means three out of four equal parts; as a quotient means three divided by four; as an operator means a scalar or three-quarters of a quantity; as a ratio signifies three objects to four objects, where the objects are of different categories; and finally, as a measure represented by iterating the unit fraction, 1/4, three times on a number line. Kieren (1980) asserts that learners must understand and function with these interpretations as prerequisites for having complete, mature knowledge of fractions. Other researchers view that learners' difficulties stem exactly from what seems like perplexing, overlapping ideas about fractions (Ohlsson, 1988).

In the usual fraction taxonomy, the interpretations or "sub-constructs" share partitioning as their foundational cognitive action. As Kieren (1980) notes, "[p]artitioning is seen here as any general strategy for dividing a given quantity into a given number of "equal" parts. Thus, it can be seen as important in developing all of the five sub-constructs." (p. 138). Positing that partitioning is the cognitive basis for fraction knowledge implies that the part/whole interpretation is the initial entry to the concept.

After this introduction to fraction knowledge, current policy and curriculum documents suggest the measurement interpretation (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Siegler et al., 2010). Following Kieren's (1980) taxonomy, they mean positioning fractions on a number line and iterating a unit fraction (a fraction whose numerator is one) to locate a non-unit fraction. For instance, starting in the third grade, the Common Core State Standards recommends that the second interpretation of fractions to study is measurement: "[r]epresent a fraction a/b on a number line diagram by marking off a lengths 1/b from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line" (p. 28). An instructional imperative for this fraction interpretation is that number lines illustrate that fractions represent magnitudes (Fazio & Siegler, 2011) and transfer to tasks involving fraction magnitude comparisons (Hamdan & Gunderson, 2017).

In the following four subsections, we discuss our distinctive measuring perspectives, define unitizing, and relate it to fraction representational models. Finally, we discuss fraction strategies.

Measuring Perspective

Contrasting with the measurement interpretation of fractions within the partitioning perspective, which concerns the equal subdivision of a single entity such as an area or a length, another measurement standpoint can also yield fractional numbers. Distinctively, this standpoint posits that a fraction represents a particular relation between two quantities of the same kind. The relation is a multiplicative comparison. We call this standpoint a measuring perspective of fractions (Powell, 2019a, 2019b) since the quantitative comparison of continuous quantities (such as length, area, volume, and time) is the cultural practice of measuring one quantity by another of its kind considered as the unit. In this perspective, measuring is the material source of both whole numbers and fractions (Davydov & Tsvetkovich, 1991). In fact, rather than fractions being an extension of whole numbers, whole numbers arise as a special case of measuring. For example, to find the extent of a distance d, in comparison to a unit of measure u, there are two cases: Either d equals an exact multiple of u, or it does not, which historically occasioned the ideas of both rational and irrational numbers.

While describing the Elkonin-Davydov curriculum, Davydov and Tsvetkovich (1991) argue for a general concept of numbers based on measuring magnitudes as the support for learning about integers and real numbers. Numbers result from the count of the iteration of a unit when measuring a magnitude. Measuring occurs when objects have a common attribute that can be compared such as length, area, volume, or mass. Then, for example, a unit of length measuring the length of an object results in a count or number. The Elkonin-Davydov curriculum asks students first to consider the relationship between the unit's size and the measure obtained when comparing it to a quantity and then to notice that the measure of the quantity decreases as the unit of measure increases. This relationship is an essential idea for understanding inverse proportionality and fractions. Fractions are introduced by accepting partial units. Overall, as Schmittau (2005) notes, Davydov's approach intertwines three design elements: "initial development from the most generalized conceptual base, ascent from the abstract to the concrete, and appropriation of psychological tools" (p. 16). Our work incorporates these three curricular design elements. First, as psychological tools, we use Cuisenaire rods for mental models (see Figure 1) and engage relational thinking with them. Second, as the generalized conceptual basis for understanding units and fractions, we measure non-discretized quantified linear quantities. Finally, we structure a learning progression by starting with fractions of quantity and then moving to fractions as numbers. With these design elements within the measuring perspective, we provided examples of fraction tasks in the Methods section.

Figure 1

Cuisenaire rods, ten different sizes and colors, arranged in a "staircase" formation.

Our study aimed to instantiate an exclusive approach, comparing two distinct quantities of the same kind (Vergnaud, 1983), with a linear model employing Cuisenaire rods (Cuisenaire, 1952). Gattegno (1974/2010) references these manipulatives (see Figure 1) as he summarizes the role of measurement for elementary mathematics:

> *Measure, in the work with the rods, is borrowed from physics and introduces counting by the back door, since it is necessary to know how many times the unit has been used to associate a number with a given length. But measure is also the source of fractions and mixed numbers, and serves later to introduce real numbers. Thus measure is a more powerful tool than counting, which it uses as a generator of mathematics. Counting ... can be interpreted again as being a measure with white rods. Measure is naturally also an interpretation of iteration (p. 196, original emphasis)*

Using Cuisenaire rods to model fractions, Gattegno views length as their attribute of interest and to be measured. His approach is exclusive in that it ventures to find "how many times the unit has been used to associate a number with a given length," so the unit and the given rod are two distinct entities with length as their common attribute. The approach is consistent with the measuring perspective. The measurable characteristic of Cuisenaire rods is one reason we chose it to engage the PSTs in reexamining how they understand fraction magnitude, order, equivalence, and operations.

Unitizing

The concept of unitizing transcends the borders of the partitioning and measuring perspectives. Nevertheless, within each view, unitizing involves a different number of quantities. Fundamentally, unitizing concerns assigning a given quantity as a unit of measure (Lamon, 1996, 2007). For example, in the partitioning perspective, unitizing is a process alongside dividing and distributing equally:

> *Partitioning is an operation that generates quantity; it is an experience-based, intuitive activity that anchors the process of constructing rational numbers to a child's informal knowledge about fair sharing. Unitizing is a cognitive process for conceptualizing the amount of a given commodity or share before, during, and after the sharing process. (Lamon, 1996, p. 171)*

Moreover, Lamon (2012) emphasizes that unitizing is both natural and subjective. For instance, given a chocolate bar segmented into eight pieces, if a child wishes to share it fairly among herself and three other children, she must decide how to divide it into sizes or unitize the bar. The child has several choices for the unit. One possibility is that she selects the unit as two segments of the chocolate bar. In this case, each child receives one whole unit of chocolate or one-fourth of the bar. Instead, she might choose each segment as the unit, and, therefore, each child will receive two units of chocolate or two-eighths of the bar. In both distribution scenarios, as two-eighths and one-fourth describe equal portions of the chocolate bar, they are equivalent fractions. The child's sharing is seen as natural, and how to size the shared pieces or unitize them is subjective. Furthermore, unitizing in different ways can yield equivalent fractions. It is worth underscoring that unitizing in the partitioning perspective is a cognitive action on a single quantity.

In contrast, in our measuring perspective, two related but distinct material quantities, physical or mental, are necessary. Further, the idea of unitizing depends on the specific meanings of the concepts of measurement and measuring. Measuring requires two quantities, the one whose extent needs to be quantified and the quantity whose size is the measuring unit. A measurement quantifies a quantity's extent, a value representing how much it is of a given unit of measure. Measuring is the action to determine the size of a quantity. Unitizing, the choice of unit of measure, is contingent. For example, a person can choose to measure the distance between two cities, using a person's natural stride as the unit of measure or a more extended quantity. The choice is subjective. For instance, in Figure 2, we present the measuring action with Cuisenaire rods, where the tan rod's length measures eight white rods, four red rods, or two purple rods. Its length depends on whether we chose the white, red, or purple rod as the unit of measure. If the orange rod (equal to 10 white rods and five red rods) is the unit of measure, then the tan rod is either eight-tenths or four-fifths of the orange rod, contingent on whether the white or red rod is the subunit. The choice is an instance of unitizing, assigning quantities as the unit and subunit of measure.

Figure 2a
The length of the tan rod measured by white, red, and purple rods.

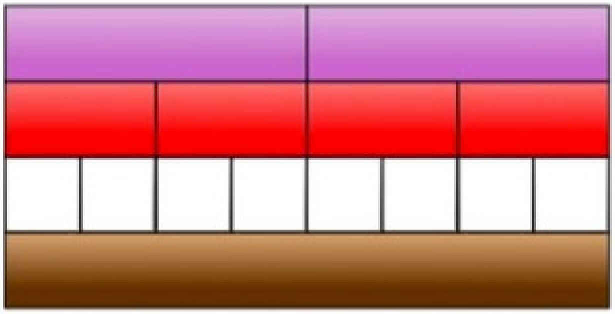
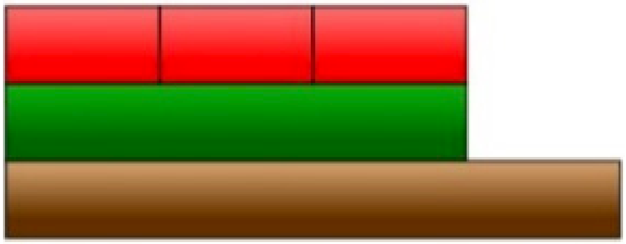


Figure 2b
The length of the dark green rod measured by red rods and compared to the tan rod.



Unitizing also pertains to determining the unit to measure a given quantity. For example, in Figure 2b, if we consider the length of a dark green rod to be three-fourths, then to unitize means to find what length is the unit of measure. In this case, since the dark green rod equals three red rods, and each red rod is one-fourth of the tan rod, then the measuring unit is the tan rod. The length of the dark green rod is three-fourths of the length of the tan rod. Since unitizing is subjective, three-fourths can also be measured by a different unit of measure. For example, the purple rod can be the unit of measure. It measures four white rods, and the light green rod measures three white rods, which means that the light green rod is three-fourths of the purple rod. That is, three measured by four is three-fourths. The need to unitize occurs when comparing two or more quantities, each measured by a different measuring unit.

Overall, unitizing is a critical operation for working adeptly with fractions. It is fundamental to fraction comparisons and operations (Van Ness & Alston, 2017a, 2017b, 2017c). Fraction comparisons, addition, subtraction, and division require that the involved quantities have the same unit of measure. For multiplication, the unit of measure of one fraction needs to equal the number of units of the other fraction's unit of measure. From the measuring perspective, in the Methods section, we illustrate comparing, adding, and subtracting fractions.

Fraction Representational Models

Mathematical representations are considered an essential element of mathematical knowledge. National standards call for supporting students to engage "in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (National Council of Teachers of Mathematics, 2014, p. 10). Representations are "processes and products that are observable externally as well as to those that occur 'internally,' in the minds of people doing mathematics" (National Council of Teachers of Mathematics, 2000, p. 67). They play a critical role in mathematics instruction (Watanabe, 2002). Policy and curriculum documents expect teachers to employ models that support students understanding and communicating mathematical ideas (Ball et al., 2008).

When teaching fractions, teachers rely on various models to illustrate the fraction concepts. Models include area, length, and sets. The area model uses geometric shapes such as circles and rectangles divided into equal parts. This model is widely used in textbooks and corresponds to fractions' part/whole interpretation (Hodges et al., 2008). The circular area model (or pies) is also commonly used to introduce fractions. However, research indicates that this model causes students difficulties partitioning circles into equal parts, especially with many equal parts (Cramer et al., 2002). The length models such as fraction strips, number lines, or Cuisenaire rods involve comparing or partitioning lengths. As mentioned above, the length model, specifically the use of number lines, is recommended by U.S.-based national organizations. Number lines support students to develop an understanding of fractions magnitudes. The set model involves comparing discrete collections of objects such as colored counters. Students consider a specific number of objects as the unit and use that number to name another group of objects' fractional relation to the unit.

Manipulatives and visual models also correspond to actions performed to identify or represent fractions. For example, Vergnaud (1983) and Watanabe (2002)

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identify two methods of representing fractions: inclusive or part/whole and exclusive or comparison. The inclusive approach uses one quantity (area, length, or a set of objects) to represent the whole and its parts. The exclusive approach involves two quantities; one quantity is the whole or unit used to compare the second. In an inclusive approach, the whole and the parts of interest are of the same object—for example, an area model represented with pizza results in comparing pizza slices to the whole pizza. The exclusive approach involves comparing two different entities of the same kind. For instance, a student compares a ruler's length to the length of a table's side. The two objects are comparable since they share a common attribute, length. To compare, the student unitizes and measures, which are both actions that correspond to the fraction concept's origin (Davydov & Tsvetkovich, 1991).

Choosing an inclusive or exclusive approach depends on the desired reasoning to be exercised. For example, when presented with partitioned parts, students use additive reasoning to count the highlighted parts and the total number of parts in the whole to identify fractions. However, this additive reasoning impedes proportional reasoning (Mack, 1995; Ni & Zhou, 2005). To develop proportional reasoning, "students must move from an additive method of comparing to a multiplicative one" (Vergnaud, 1983, p. 162). Multiplicative reasoning engages students with language that focuses on the scaler relation between quantities such as "how many times more or less."

In contrast, additive language reflects counting procedures and uses statements like "two or three objects more or fewer." This difference between additive and multiplicative reasoning highlights the importance of engaging students with continuous quantities when dealing with fractions. Continuous quantities are quantities for which there is another measure between any two measures (e.g., length, area, volume, or time). With any two non-discretized continuous quantities, students engage in unitizing, measuring, and comparing the two quantities to identify the fractional relationship between them. For example, Maher and Yankelewitz (2017) found engaging students with measuring and comparing lengths (Cuisenaire rods) to identify and compare fractions supports them to reason successfully about fractions (indirect, using cases, counterargument, and recursive reasoning) and to transition from additive to multiplicative reasoning.

Fraction Strategies

How learners resolve tasks involving fractions has been the focus of numerous studies. Several studies looked at how students compare fractions. Among these studies, a common finding relates to what has been called the whole number bias, strategies in which students employ properties of whole numbers to fractions. An example is Mack (1990), which we discussed in the introduction. Similarly, Erol (2021) asked fifth-grade students to compare fractions and then interviewed them to understand their reasoning. Relying on whole numbers properties to determine the greater of two fractions, students stated that a fraction is greater (1) when the numerator is larger among fractions with the same denominator, and (2) the fraction with the larger denominator is the greatest among fractions with the same numerator. Gabriel et al. (2013) observed this latter strategy for comparing fractions with fourth, fifth, and sixth-grade students. In two experiments, Meert et al. (2013) tested college students' componential and holistic processing of fraction comparisons. They asked the students to compare fractions with and without common components. When comparing fractions with the same denominators, college students only compared the magnitude of the numerators. With fractions having the same numerators, they used holistic processing to compare them.

Researchers observed other fraction strategies when learners located fractions on number lines. Siegler et al. (2011) identified two strategies, numerical transformation and segmentation. Zhang et al. (2017) subsequently investigated these strategies with middle school students who located fractions on 0-1 and 0-5 number lines. They found that only a few students used numerical transformation strategies by converting the fraction to a decimal or rounding it and comparing it with 0,1/2, and 1 as benchmarks and converting an improper fraction into a mixed number. Students used segmentation strategies accurately by dividing the number line into equal parts corresponding to the value of the fraction's denominator. However, they used segmentation strategies inaccurately by segmenting the unit into unequal intervals. When locating fractions on a 0-5 number line, some students treated the 0-5 number line as if it were a 0-1 number line, for example, by locating 7/8 close to 5. This action suggests that they think fractions are always less than one. Similarly, Bright et al. (1988) found that fourth and fifth-grade students had difficulties identifying the unit when locating fractions on a number line. Students counted the marks to locate fractions, which did not correspond to the equal interval related to the fraction.

Research about fraction strategies related to representational models such as set, rectangular, or circular models is limited. Additionally, there is scarce research about instructional interventions from a measuring perspective. Therefore, our study examines the strategies that PSTs used with these three models before and after revisiting fractions from a measuring perspective.

Methods

The intervention engaged pre-service elementary teachers in reexamining fractions from a measuring perspective to investigate changes in how they reason about fractions using different representational models of fractions. We use "reexamining" to indicate a re-consideration of PSTs' understanding of fractions, including a part/whole conception, from a measuring perspective. The intervention took place in an elementary mathematics methods course in a 15-week semester. During the first week of the semester, the PSTs completed a pre-test in which they expressed fractions using discrete and continuous models. In the last week of the semester, PSTs completed a similar assessment as a post-test. For approximately 75 minutes every two weeks, the PSTs used Cuisenaire rods (see Figure 1) to collaboratively solve fraction tasks.

The tasks initially engaged them with whole numbers and operations problems to familiarize the PSTs with Cuisenaire rods. Afterward, a set of tasks introduced them to fractions by comparing the lengths of different rods multiplicatively. The tasks did not include situations involving two-dimensional rectangular, circular, or set fraction models. Instead, PSTs interacted with three-dimensional physical objects formed by six parallelograms or parallelepipeds (Cuisenaire rods), focusing on one of their dimensions, length. An outline of the intervention tasks follows a description of the study participants.

Participants

The participants were 46 pre-service elementary teachers (43 females) enrolled in an elementary mathematics methods course at a medium-sized state university in the northeast of the United States. This study's participants consist of PSTs from two sections (n=22 and 24) of the course during the second semester of 2017. The participants were in their last year of a four-year early childhood baccalaureate degree program and one semester away from student teaching. In the program's first year, they completed two mathematics content courses specifically designed for pre-service elementary teachers and covered topics from elementary school mathematics, including rational numbers, ratio, and proportion.

The elementary mathematics methods course focused on problem solving and mathematical reasoning. It discussed the design and implementation of mathematical tasks that support elementary students to develop a conceptual understanding of various mathematical topics. The topics included the development of whole-number sense, operations on whole numbers, geometry, probability and statistics, early algebraic ideas, and fractions, including adding and subtracting fractions. For lack of time, the course did not include the multiplication and division of fractions. The course met for two 75-minute sessions a week during 14 instructional weeks. Starting the third week of the semester, the intervention for this study consisted of four sessions. Participants solved fraction tasks collaboratively, using Cuisenaire rods, every two weeks for an entire session. These fraction sessions represented approximately 14% of the semester. The sessions were staggered to lessen the cognitive load and give the participants extended time to reflect on their learning.

Reexamining Fractions from a Measuring Perspective

What follows is an outline of how we invited the PSTs to use Cuisenaire rods to rethink their fraction knowledge from a measuring perspective.

1. Familiarization with Properties and Operations: PSTs play with rods and note their properties. We define a train of rods as one or more rods placed end-to-end and have PSTs create trains. They construct trains and compare absolute lengths that go beyond the available rod sizes. For instance, the teachers build trains such as an orange and a purple train and compare its length to a train consisting of three rods: yellow, purple, and green. Next, they add and find the difference between pairs of lengths. Finally, they multiply lengths by iterating a chosen unit length a desired number of times and dividing lengths by seeing how many units create a length congruent to a larger length. These experiences enact a measuring perspective with whole numbers.

2.Introducing Fractions: From defining and iterating a unit to obtain a certain length, PSTs compare the length of the unit and the resulting length. For example, they repeat the red rod four times to create a train equivalent to the brown rod (see Figure 2a) and say that the length of the brown rod is four red rods. The inverse relation between the original unit and the resulting length yields a unit fraction. In the above example, the brown rod becomes the unit, and the red rod becomes the rod to be measured, whose length is one-fourth of a brown rod. Then, PSTs compare two red rods, five red rods, or 10 red rods to the brown rod and say that it is two-fourths, five-fourths, or ten-fourths of the brown rod, respectively. These comparisons lead participants to name non-unit and improper fractions. They also measure rod lengths by iterating a unit length where a whole number of unit rods does not create a length congruent to a larger length. For example, they measure the black rod (7 cm) using a light green rod (3 cm) as the measuring unit. They need two light green rods and a white rod to create a length equal to the black rod (see Figure 3). The name of a white rod's length emerges from the inverse relationship between its length and the length of a light green rod (one light green rod equals three white rods, and one white rod equals one-third of a light green rod). As shown in Figure 3, the measure of the length of the black rod is two and one-third of the light green rod.

Figure 3

Measuring a black rod using a light green rod, which measures 2 and 1/3 light green rods.

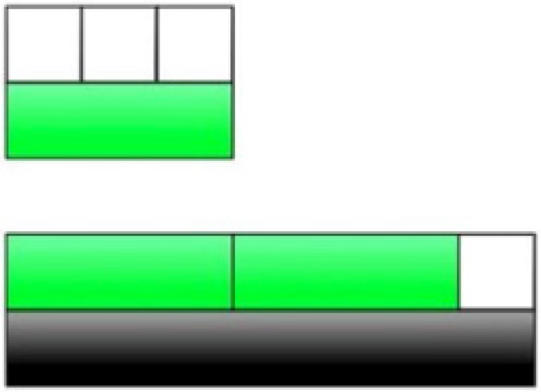
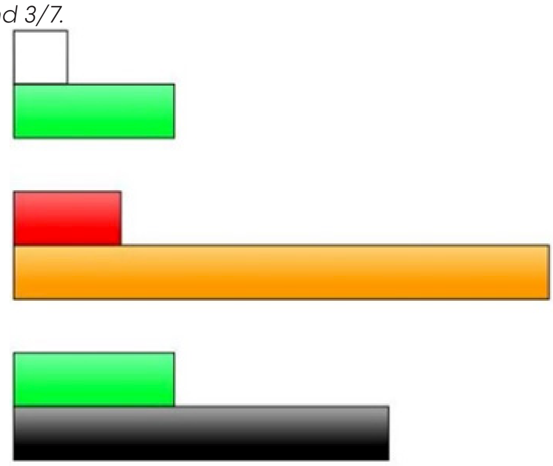


Figure 4

Using Cuisenaire rods to illustrate the fractions 1/3, 1/5, and 3/7.



Afterward, PSTs compare pairs of rods and name the fractions that represent the comparisons. Examples are that the white rod is one-third of the light green rod, the red rod is one-fifth of the orange rod, and the light green rod is three-seventh of the black rod (see Figure 4). They also compare the lengths of different rods to the lengths of trains of rods. For example, they compare the brown rod's length to a train's length composed of an orange rod and a yellow rod. PSTs then express the multiplicative comparison as eight-fifteenths since the brown rod is 8 cm and the train consisting of an orange rod and a yellow rod is 15 cm. We do not explicitly use a standard unit of measurement (centimeters or inches) to compare lengths; we use other rods such as the white rod (1 cm) or the red rod (2 cm) as an intermediary or subunit to assist with comparing lengths.

3. Comparing fractions: PSTs think of fraction magnitudes such as one-half and one-third and decide which is greater. They then demonstrate their choice using the rods. Some of them create those pairs of rods to model the fractions without regard to a standard unit of measure and then compare them. They create any of these three situations:

a) one-half is larger than one-third, using a red rod as one-half of a purple rod and a white rod as one-third of a light green rod; a red rod is larger than a white rod (Figure 5a);

b) one-half is equal to one-third, using a white rod as one-half of a red rod and also one-third of a light green rod (Figure 5b); and

c) one-half is smaller than one-third, using a white rod as one-half of a red rod, and a light green rod is one-third of a blue rod; a white rod is smaller than a light green rod (Figure 5c).

Figure 5a

Illustration of 1/2 and 1/3 where the rod representing 1/2 is larger than the rod that represents 1/3.

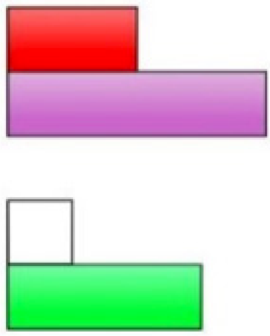


Figure 5b

Illustration of 1/2 and 1/3 where the rod representing 1/2 is equal to the rod that represents 1/3.

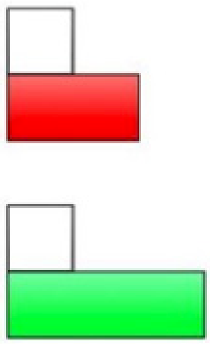
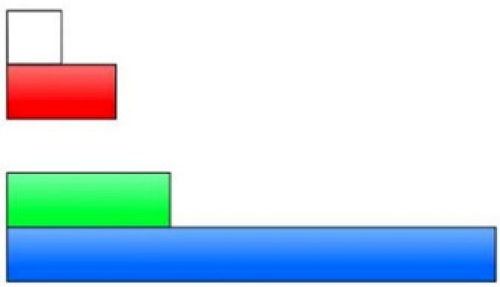


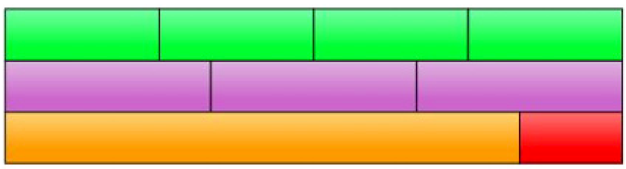
Figure 5c

Illustration of 1/2 and 1/3 where the rod representing 1/2 is smaller than the rod that represents 1/3.



These discrepant results instigate a discussion about the role and importance of a unit length of measure to represent fractions such as one-half and one-third. Afterward, PSTs use one length as the unit of measure and find the corresponding lengths that represent one-half and one-third so that they can compare their relative magnitudes. Next, they learn the Train Race game to identify commensurable unit length for a set of fractions. The game results in the length that represents the least common multiple of the lengths of two rods by placing them next to each other and creating a single-color train using the two rods until they are equal in length. That length becomes the standard unit of measure for the fraction comparisons. For example, to compare one-third and one-fourth, PSTs take the smallest rod with which they can represent thirds (light green) and the smallest rod with which they can represent fourths (purple rod) and place them side by side (see Figure 6). From those two rods, they create two single-color trains. The light green rod is shorter than the purple rod, so they add another light green rod, then compare again to see which train is shorter and add another rod to it to make it longer. In this case, they add another purple rod and continue in turn until the two trains are equal in length. This Train Race ends with four light green rods and three purple rods since the length of the two trains is now equal. The last rod added to the trains is the rod color that wins the game. The length created from using a light green rod and a purple rod allows PSTs to represent one-third (one purple rod) and one-fourth (one light green rod) and compare their magnitudes.

Figure 6
The result of a Train Race Game involving green and purple rods.

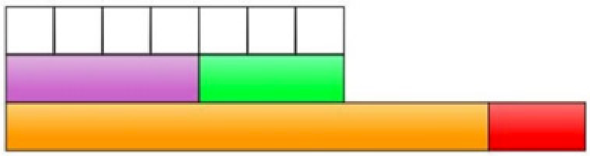


4. Adding and Subtracting Fractions: PSTs use the Train Race game to find a unit length to represent different fractions. They create a train composed of one-third and one-fourth and identify this new train's multiplicative comparison to the unit, which equals a train of an orange and a red rod. In this example, one-third is a purple rod, one-fourth is a light green rod, and the train composed of those two rods is seven-twelfths of the unit. To demonstrate subtracting fractions, PSTs identify the difference between the two lengths that represent the two fractions. For example, using the length that resulted from the Train Race above, teachers find the difference between one-third (a purple rod) and one-fourth (a light green rod). They identify the rod that fills the rod's gap when placing them side-by-side (see Figure 7).

Figure 7
Using Cuisenaire rods to illustrate the difference between 1/3 and 1/4.

In this case, it is the white rod. They express the rod (or length) that fills the gap with the unit. Now, a white rod is one-twelfth (1/3-1/4=1/12). The PSTs map this physical experience of comparing, adding, and subtracting fractions to the symbolic manipulation procedure and finding other fractions' names. In the example above, they express the fraction one-third differently, and it would be four-twelfths as a purple rod is equal to the length of a train composed of four white rods. Each white rod is one-twelfth of the unit. Similarly, they express one-fourth as three-twelfths and write 1/3+1/4=4/12+3/12=7/12 (see Figure 8). A final interrogation concerns writing this statement without representing the fractions using the rods.

Figure 8
Using Cuisenaire rods to illustrate adding 1/3 and 1/4.



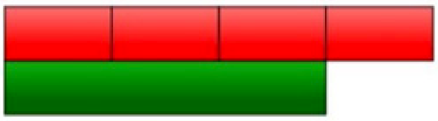
Most of the tasks above involve less-than-one fractions only to illustrate the measuring approach. In fact, each task engages participants with fractions greater than one immediately after working with fractions less than one. For example, participants measure the red rod (2 cm) using a dark green rod (6 cm) as the measuring rod and notice that one red rod is one-third of the dark green rod. Again, in relation to the dark green rod, they also measure two red rods and four red rods, respectively two-thirds and four-thirds (see Figures 9a, 9b, and 9c). The participants are encouraged to refrain from using the language of mixed numbers to name the fractions so that their language corresponds closely to what they see. After participants develop fluency with naming fractions, the language of mixed numbers is visited.

Figure 9a
Comparing one red rod to a dark green rod.

Figure 9b
Comparing two red rods to a dark green rod.

Figure 9c

Comparing four red rods to a dark green rod.



Similarly, when comparing fractions or adding and subtracting fractions, participants are asked to compare, add, and subtract greater-than-one fractions immediately after working with fractions less than one. For example, when comparing fractions, participants are asked to determine which of these two fractions is greatest: four-thirds and five-fourths. They use the Train Race to find a unit length representing the two fractions (orange and red rods; see Figure 10). The unit length is equivalent to the train in Figure 4 since the comparisons involve thirds and fourths. Participants can observe that the train representing four-thirds (four purple rods) is longer than the train representing five-fourths (five light-green rods), which means four-thirds are greater than five-fourths. PSTs discuss the difference between the two lengths that represent the two fractions. They can see that the difference between the two trains (four-thirds and five-fourths) is one white rod, filling the gap between the trains. They express the white rod's name in relation to the unit as one-twelfth (see Figure 10), demonstrating the subtraction of four-thirds and five-fourths. When asked about the sum of four-thirds and five-fourths, participants use the two trains that represent the two fractions to create one train composed of those two trains, then compare this new train to the unit. They can identify that the new train is equivalent to 31 white rods, thirty-one-twelfths. Participants can also notice that the new train equals two trains of the unit and seven white rods (two and seven-twelfths).

Figure 10

Using Cuisenaire rods to illustrate comparing and subtracting 4/3 and 5/4.

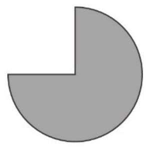
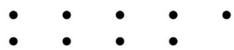
With this measuring perspective for reexamining fractions, participants compared any two quantities and choose an appropriate unit of measure. For example, the orange rod can be the unit of measure, making the yellow rod one-half, the purple rod four-tenths or two-fifths, and the white rod one-tenth. When the yellow rod is the unit of measure, the orange rod is ten-fifths, the purple rod is four-fifths, and the white rod is one-fifth. The PSTs continually consider the quantity to be measured to determine a unit and often a subunit of measure.

Data Collection and Analysis

This study's data come from pre- and post-tests that participants completed in the first and last weeks of the semester about aspects of their fraction knowledge. We adopted Norton and Wilkins's (2010) fraction assessment to examine PSTs' facility with unitizing and representing fractions less than one and greater than one, using two different continuous models (rectangular and circular) and a discrete model (set of dots; see Table 1). Each test includes 10 items that involve only two of the fractions' models, namely, dots and circles, dots and rectangles, or circles and rectangles. The fraction questions are parallel among the three representations. We randomly assigned participants to one of the three pre-assessment versions. For the post-test, we ensured that each participant received a different version of the assessment and answered items that used each of the three formats.

Table 1

Pre- and post-tests item samples. Adapted from (Norton and Wilkins 2010)

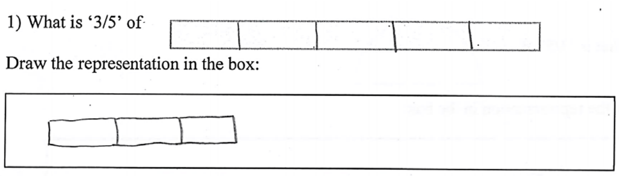
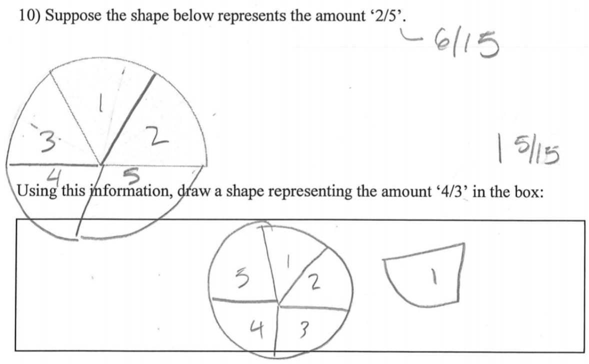
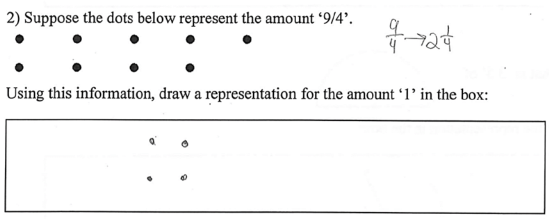
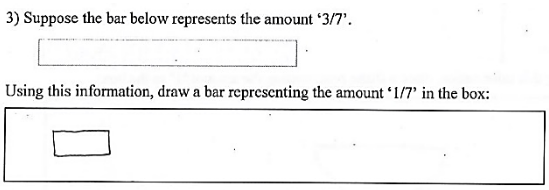
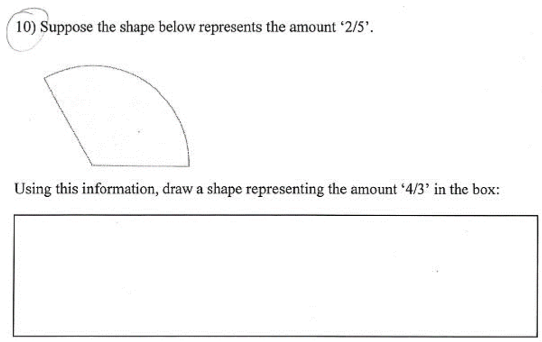
Circular model	7) Suppose the shape below represent the amount '9/4'.  Using this information, draw a shape representing the amount '1' in the box.
Rectangular model	7) Suppose the bar below represents the amount '9/4'.  Using this information, draw a bar representing the amount '1' in the box.
Set model	7) Suppose the dots below represent the amount '9/4'.  Using this information, draw a representation for the amount '1' in the box.

We conducted a conventional content analysis (Hsieh & Shannon, 2005) to identify PSTs' strategies for solving fraction problems. For multiple iterations, two researchers coded teachers' responses and discussed the codes until they agreed on a set of codes (see Table 2). After that, each researcher coded the same 240 responses separately and agreed on 257 codes out of 279, or 92.11% agreement.

Results

Our coding of PSTs' responses to the pre- and post-tests revealed five strategies for solving the fraction tasks described above, involving the set, rectangular, and circular fraction models. The tasks invited PSTs to identify (a) a fractional relation between two quantities or (b) the portion of a quantity that

Table 2
Definitions and examples of fraction strategies from pre- and post-tests

Partitioning: PSTs partition a quantity (set, length, or area).	
Constructing: PSTs draw an extension to a quantity (set, length, or area).	
Symbolic manipulation: PSTs create equivalent fractions or change improper fractions to mixed numbers.	
No visible strategy: PSTs provide an answer without showing any work.	
No answer: PSTs do not provide any written response.	

represents a certain fraction. Table 2 presents the five strategies and an example of each.

Generally, the frequencies and percentages of the strategies that PSTs expressed in their responses on the pre- and post-tests shifted. Our analysis revealed that the number of codes for strategies increased on the post-test from 527 to 578 codes. The percentages for partitioning strategy compared to all strategies increased from 35% in the pre-test to 52% in the post-test. In addition, fewer responses on the post-test included no visible strategy; its percentage decreased to 24% in the post-test from 40% in the pre-test. These findings indicate that compared to the pre-test participants employed more strategies on the post-test. In the following, we present PSTs' strategies for tasks involving circular, set, and rectangular models and fractions less than and greater than one.

Strategies Related to the Circular Model

In another study (Alqahtani & Powell, submitted), we scored the accuracy of PSTs' responses. For each response, the score was either 0, .5, or 1. For tasks that involved the circular model, Table 3 shows the percentages of each strategy associated with PSTs' responses and the mean score for the accuracy of their responses.

Table 3

Percentages of codes and mean score for the circle representational model.

Strategy	Pre-Test		Post-Test	
	%	Mean score	%	Mean score
Partitioning	55%	.74	66%	.80
Construction	6%	.89	6%	.69
Symbolic Manipulation	17%	.37	14%	.63
No Visible Strategy	15%	.67	14%	.42
No Answer	7%	0	1%	0
Total	100%	.63	100%	.69

As shown in Table 3, the most used strategy is partitioning in both the pre- and post-tests (55% and 66%, respectively) with a slight increase in the accuracy of the responses. The construction strategy did not differ between the two test times, but the accuracy decreased from a mean score of .89 to .69. The symbolic manipulation strategy decreased usage from the pre-test (17% to 14%) and increased the mean score (.37 to .63). Almost the same number of responses had no visible strategies but decreased the mean score, going from .67 in the pre-test to .42 in the post-test. There was a noticeable change in the number of responses that had no answer in the post-test, coming down to only 1% in the post-test from an initial 7% in the pre-test.

The change in the accuracy with the responses that had construction strategy might indicate that this strategy dose not lead to accurate estimations. When PSTs draw extensions to circular sectors, they cannot compare areas accurately and identify fractional relationships. The change with partitioning strategy might indicate that this strategy is more effective when dealing with circular sectors.

Strategies Related to the Rectangular Model

Similarly, partitioning strategy was the most common with PSTs' responses to questions that involved rectangular models in both pre- and post-tests (with 50% and 78% respectively; see Table 4). A more noticeable change was the decrease in the absence of strategies. In the pre-test, about 29% of the codes were for "No Visible Strategy." That percentage decreased to 9% in the post-test, and the mean score for accuracy of responses also decreased, from .66 to .5. The PSTs answered more questions on the post-test in comparison with the pre-test. Table 4 also shows that few PSTs used constructing strategy with rectangular shapes on post-test, while none of the PSTs used this strategy on the pre-test.

Table 4

Percentages of codes and mean score for the rectangular representation model.

Strategy	Pre-Test		Post-Test	
	%	Mean score	%	Mean Grade
Partitioning	50%	.69	78%	.71
Construction	0%	-	3%	1
Symbolic Manipulation	4%	.3	9%	.5
No Visible Strategy	29%	.66	9%	.5
No Answer	16%	0	2%	0
Total	100%	.52	100%	.65

The data in Table 4 indicate that PSTs reasoned more effectively on the post-test (more answers and more strategies). This change might be related to the similarity between the materials used in the intervention and the rectangular model. That is, like interacting with rods, the PSTs may have focused on the length of the rectangles. Findings show that working with a measuring approach can improve how PSTs partition rectangular shapes to compare them and identify fractional relationships among them.

Strategies Related to the Set Model

In our analysis, the most common code for PSTs' responses to questions that involved a set model was "No Visible Strategy." The percentages of no visible strategy were 61% on the pre-test and 70% on the post-test without any change in PSTs' accuracy. They used fewer symbolic manipulation strategy on the post-test than the pre-test, while their accuracy increased. In addition, PSTs provided more answers on the post-test than the pre-test.

Table 5

Percentages of codes and mean score for the set model.

Strategy	Pre-Test		Post-Test	
	%	Mean score	%	Mean score
Partitioning	15%	.78	16%	.8
Construction	2%	.6	2%	.7
Symbolic Manipulation	16%	.61	10%	.8
No Visible Strategy	61%	.75	70%	.75
No Answer	7%	0	2%	0
Total	100%	.68	100%	.73

With all strategies, the mean scores for the accuracy of PSTs' responses increased on the post-test. The change in the frequency of symbolic manipulation strategy (decreasing from pre-test to post-test) and the response accuracy (increasing from pre-test to post-test) might indicate that this strategy is ineffective with the set model. Data did not clearly show which type of strategy is more appropriate for set model questions. Nevertheless, the improvement in accuracy shows that revisiting fractions using a measuring approach can influence how PSTs solve set model questions.

Strategies Related to Fractions Less than and Greater than One

When examining the strategies that PSTs implemented with questions that involved less than and great than one fraction, we found that the largest increase occurs with partitioning strategy (see Table 6). Partitioning strategy comprised 43% of strategies used on the pre-test and 65% on the post-test for less-than-one fractions. The response accuracy did not have a notable change. Similarly, percentages for partitioning strategy with greater-than-one fractions increased from 24% to 43%, with a sizeable mean score increase from .46 to .57. In addition, the number of responses with no visible strategy decreased from 49% to 28% for less-than-one fraction questions and from 29% to 22% for greater-than-one fractions questions. The change for symbolic manipulation strategy was marginal for responses to both types of fraction questions. However, the accuracy of responses increased mean from the pre-test to the post-test. A more noticeable finding is the change in the number of questions that received no response on greater-than-one fraction questions. On the pre-test, 18% of codes were for questions that received no answers compared to only 3% on the post-test.

Table 6 above presents a few interesting findings. Strategies that PSTs implement vary depending on the type of questions. Understandably, PSTs used more symbolic manipulation strategy with fractions greater than one, including changing improper fractions to mixed numbers. In addition, the increase in the partitioning strategy for both types of questions, along with the increase in accuracy, indicate that working with fractions from a measuring perspective can support PSTs to reason visually through partitioning. Findings also show no notable difference between the construction strategy that PSTs used for both types of fraction questions.

Discussion

This study engaged 46 PSTs in reexamining fractions from a measuring perspective and investigated their strategies to compare quantities and identify fractional relations among them. On pre- and post-tests, PSTs worked with discrete and continuous quantities, presented in three models: set, rectangular, and circular. The set model involved a collection of dots, the rectangular model involved rectangles with a fixed width, and the circular model involved circular sections. The pre- and post-tests invited PSTs either to identify the fractional relation between two quantities of the same kind or to draw a set, rectangle, or circular section representing a certain fraction of a given set, rectangle, or circular area. The intervention employed Cuisenaire rods and engaged PSTs to compare the lengths of different rods to identify fractional relations between pairs of them and add and subtract fractions. Qualitative analyses show that PSTs implemented three main problem-solving strategies with the representational models: partitioning, constructing, and symbolic manipulation. Interestingly, two strategies were similar to findings from (Siegler et al., 2011), where participants used segmentation and numerical transformation strategies.

Our analyses also revealed changes in the frequencies of strategies from the pre- to the post-tests. On the post-test, PSTs used more strategies and answered more questions. Specifically, they used more partitioning strategy with the continuous models (rectangles and circular sectors). Furthermore, when analyzing PSTs' responses based on the type of fractions involved, findings also revealed that, in the post-test, partitioning strategy increased with fractions less than and greater than one.

Our findings show pronounced changes in strategies with the rectangular model. We believe this occurs

Table 6
Percentages and mean score of the pre- and post-tests for questions less than one and greater than one.

Strategy	Less than one				Greater than one			
	Pre		Post		Pre		Post	
	%	Mean score	%	Mean score	%	Mean score	%	Mean score
Partitioning	43%	.85	65%	.84	24%	.46	43%	.57
Construction	3%	.89	4%	.78	2%	.6	4%	.67
Symbolic Manipulation	3%	.89	2%	1	28%	.44	29%	.57
No Visible Strategy	49%	.90	28%	.85	29%	.35	22%	.31
No Answer	2%	0	<1%	0	18%	0	3%	0
Total	100%	.86	100%	.84	100%	.34	100%	.5

because of the close relationship between the rectangular model and the manipulative materials used in the intervention. The measuring approach with Cuisenaire rods asks learners to measure the length of one rod using another and express that measurement using rational number. Learners investigate and decide on an appropriate unit of measure and a subunit if needed. The rectangular model questions on the assessment ask PSTs to compare rectangular shapes with a relatively small width. PSTs only attended to the lengths of the rectangle and kept the width constant. In a related study, researchers first observed this close relationship between the rectangular model and the measuring approach (Alqahtani & Powell, submitted). In that study, the authors investigated the changes in PSTs' fraction knowledge after reexamining fractions from a measuring perspective. The participants from the current study comprised about half of the previous one. In general, the scores of 96 PSTs (including the 46 participants from the present study) on the post-test show a statistically significant increase (at $p < 0.01$) compared to the pre-test. With questions that involved greater-than-one fractions and for each of the three representational models, the scores also increased significantly (at $p < 0.05$ for the rectangular and circular models and $p < 0.01$ for the set model). With questions that involved fractions less than one, the authors found that participants' scores show statistically significant increase (at $p < 0.05$) only with the rectangular model. Again, the similarity between the rectangular model and the intervention's manipulatives might explain this change.

Even though PSTs did not work with the part/whole perspective or the partitioning action during the intervention, results show an increase in partitioning strategy with the rectangular and circular models. We believe that operating on continuous quantities such as length and area to compare and identify fractional relations is a conceptual process. The partitioning strategy that PSTs employed involves measuring or estimating the magnitude of lengths or areas. PSTs used partitioning to measure or estimate the size of the unit or the unit fraction. Working with Cuisenaire rods may have supported the PSTs to unitize, compare absolute magnitudes, and identify relative magnitudes between two continuous quantities. In alignment with Sengul (2013) and Yang et al. (2009), we contend that partitioning strategy based on measuring and estimating quantities reflects a conceptual understanding of fractions.

This study contributes to the literature by analyzing the implementation of a measuring perspective for fraction learning using Cuisenaire rods. This perspective aligns with the theoretical position and empirical studies that measuring is the material source of both whole numbers and fractions (Davydov & Tsvetkovich, 1991; Gattegno, 1974/2010). Another contribution of this study is the discussion of strategies that PSTs employ when working with fractions represented in three different models (set, rectangular, and circular) before and after revisiting fractions from a measuring perspective. The three models allowed PSTs to engage with counts, lengths, and areas. Conceptual strategies, such as measuring-based partitioning instead of counting-based partitioning, may support PSTs' fraction knowledge. The other two strategies, symbolic manipulations and construction, did not seem adequate for comparing two quantities and identifying fractional relations between them.

Future research may examine the influence of learning fractions using a measuring approach with both elementary and middle school students. Research is also needed to study how individuals use the three fraction models with fraction arithmetic and investigate how that compares to strategies used by those who worked with Cuisenaire rods within a measuring perspective.

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