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Determining Ways of Thinking and Understanding Related to Generalization of Eighth Graders*

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Abstract

The main purpose of this study is to determine ways of thinking and understanding of eight graders related to generalizing act. To carry out this aim, a DNR based teaching experiment was developed and applied to 9 eight graders. The design of the study consists of three stages; preparation process in which teaching experiment is prepared, teaching process in which teaching experiment is applied, and analysis process in which continuous and retrospective analyses are carried out. Analysing the data, it was found that students' ways of thinking could be determined as relating, searching, and extending. Ways of understanding belonging to generalizing act could be determined as identification, definition, and influence. It was recommended to add two new categories "relating with an authority" and "searching the same piece" to the generalization taxonomy.

Keywords: Based Instruction, Generalization Taxonomy, Ways of Thinking, Ways of Understanding

Introduction

Learning occurs as a result of the equilibrium of one's mind. Mathematical knowledge, as well, is the product of this balancing process. Many researchers, despite often using different terminology, highlight the operative and conceptual structure of mathematical knowledge (Hiebert & Lefevre, 1986; Sfard, 1991; Skemp, 1976). One might argue, then, that mathematical knowledge and, therefore, mathematics itself are composed not just of axioms, definitions, and theories, but also of the conceptual tools that allow our minds to access this knowledge (Harel, 2008a).

Mathematics teachers are concerned with how students can improve their ability to learn and adopt different approaches in the classroom. DNR-based instruction (DNR is the acronym of duality, necessity and repeated-reasoning) is a teaching approach that seeks to provide answers to challenges which mathematics teachers continue to face. To do so, this approach focuses on improving ways of thinking and understanding of our mental acts. Problems are therefore perceived in a manner that considers students' intellectual needs as well as their willingness to solve problems. When students learn desirable ways of thinking and understanding, it is necessary to make this information permanent by repeated reasoning so they can integrate them into their academic lives.

Since the student and the student's knowledge formation processes are at the centre of teaching process, it is necessary to try to understand what is going on in their minds while investigating their generalization processes. Generalization strategies developed in the literature approach these processes from the perspective of a researcher. Based on actor-oriented transfer, Ellis (2007) investigated the generalization processes of students and interpreted this process from the perspective of the student. Additionally, the generalization process was examined as a process and product in line with the structure of mathematics, and the generalization taxonomy was developed. The aim of this study was to determine the ways of thinking and understanding of students while generalizing. Eighth-grade students were subjected to a DNR-based teaching experiment, and their generalization processes were analysed according to this taxonomy.

Theoretical Framework

Learning means using the knowledge (Piaget, 1964, p. 20). Learning mathematics requires the use of the acquired mathematical knowledge, and an improvement of mathematical thinking. Mathematical thinking involves many interconnected thinking styles such as geometrical thinking, functional thinking, and algebraic thinking. Algebraic thinking is related to noticing patterns and investigating the mathematical relations of numbers, objects, and geometrical shapes; therefore, they compose the basics of the mathematical thinking (Windsor, 2009). Two important components of primary school algebraic thinking are making generalizations and using symbols to show and solve problems (Carpenter & Levi, 2000).

Students intend to classify mathematical structures according to their appearances in order to make generalizations. The characteristics of generalization approaches conducted in later years involve making logical-mathematical inferences beyond their appearances, noticing relationships, and expressing these relationships with symbols. Therefore, in the process of learning mathematics and improving the ability to generalize, mathematical representation is very important (Carraher, Martinez & Schliemann, 2008). When students start making generalizations by observing patterns, algebraic thinking commences as well.

* This study has been obtained from doctoral dissertation titled as "Determining the 8th Graders Ways of Thinking and Ways of Understanding Related to Generalization: A DNR Based Teaching Experiment" completed by Gülçin Oflaz.

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Generalization is the core of algebraic thinking. Along with being an effective algebraic 'procept' (Gray & Tall, 1994, p. 95), generalization represents both process and product, is an irreplaceable instrument in the representation of mathematical modelling, problem-solving, and quantitative relations with symbols (Becker & Rivera, 2006). Indeed, generalization is understood as either product or process. However, generalization requires both product and process, as the process of generalization leads to an expression (Yerushalmy, 1993). This study regarded generalization as both a process (ways of thinking) and product (ways of understanding).

Ways of Thinking of Generalization Act

The ways of thinking of generalization act are the characteristics of one's cognitive process when one makes generalization. According to the literature, generalization strategies can be categorized as recursive thinking, explicit thinking, visual thinking, quantitative thinking, and pragmatic thinking in which both quantitative and visual thinking are included (Barbosa, 2011; Becker & Rivera, 2005; Chua & Hoyles, 2011; Lannin, 2004, 2005; Orton & Orton, 1999; Stacey, 1989; Zazkis & Liljedahl, 2002).

Recursive thinking is a technique used widely in pattern generalization problem-solving. Recursive thinking is the search for the mathematical relationship between the first term and the last term in a series. More generally, it is how a series is investigated in terms of whether the common difference between a series' sequenced terms is valid for every term in the series or not (Lannin, 2003, 2004; Stacey, 1989; Zazkis & Lildejahl, 2002). Even if recursive thinking is the first method used while finding the rule for a pattern, doing the same operation repeatedly is not sufficient. Therefore, one might argue that explicit thinking is recognized as more effective and efficient than recursive thinking.

Explicit thinking involves calculating the value of dependent variable according to the given value of the independent variable. Generally, algebra lessons given in schools involve finding a formula for a given problem. However, students may not necessarily have the required mathematical knowledge to find the correct formula. Therefore, the explicit and recursive thinking proceed as nested while finding a rule (Barbosa, 2011; Lannin, 2004, 2005).

Visual thinking is defined as explaining the shapes in a way to complete the series, even if they are not visible. It is generally used for geometrical data patterns (Friel & Markworth, 2009). Students who think visually focus on the structural properties of a shape. These students use visual images in their solutions (Becker & Rivera, 2005, 2006). For questions that require one to consider the structural characteristics of a shape, the focus can quickly move towards a "how many" question.

Becker and Rivera (2006) stated that those who prefer a quantitative approach while making generalizations use quantitative operations to find a rule. This approach requires turning the shape pattern of data into quantitative patterns and using these quantitative patterns to find a rule (Becker & Rivera, 2005; Chua & Hoyles, 2011; Tanışlı & Yavuzsoy Köse, 2011). The use of visual and quantitative thinking methods together is called pragmatic thinking (Kirwan, 2015, p. 29). Students who think pragmatically can think both visually and numerically (Becker & Rivera 2006; Tanışlı & Özdaş, 2009). Therefore, they benefit from both visual strategies and numerical strategies while finding the rule of a given pattern.

Generalization strategies developed in the literature can be mostly explained by means of the researcher-oriented transfer approach since the information regarding the previous situation to which the students establish a similarity relation cannot be obtained through such strategies. However, since the student is at the centre of the teaching process and the intention is to foster his/her knowledge of a particular topic, a classification giving more information what happens in students' mind when generalizing is needed. Ellis (2007) investigated generalization as a process (generalization actions) and product (reflection generalizations) and created her taxonomy, based on Lobato's (2003) actor-oriented approach. Lobato (2003) investigated how students form similarities among the problems encountered from learners' point of view. In this respect, Lobato (2003) shifted the transfer studies in the literature from the researcher perspective to the student perspective and added a new dimension to the transfer studies. The actor-oriented transfer gives important clues regarding how the brain establishes connections between new and old information (Lobato, 2003).

'*Generalization Actions' as Ways of Thinking*

A series of cognitive processes takes place in the brain while making generalizations. Ellis (2007) described this process as a characteristic of the generalization process in mind during the verbal or written action of a person, and classified this as 'generalizing actions'. For example, while a student is finding the rule of a pattern, s/he can solve the problem by 'relating' it to a pattern that s/he has seen previously or can solve by 'searching for the same pattern', or by making generalizations. Ellis (2007) identifies three categories of generalization approaches: relating, searching, and extending. These categories are interconnected and occur simultaneously. This classification was formed considering students' focus.

Students relate, or make connections between two or more events, situations, or problems when identifying pattern rules. The cognitive actions that take place while relating are random rather than purposeful. While looking for connections, the student is not aware that the relationship has already been established. When a student searches, s/he may repeat actions in order to identify similarities. The actions performed here are more purposeful. The student, aware of the possibility of similar relations, searches actively for similarities among many samples. If the student not only realizes the presence of a similarity relation, but s/he also explains with a more general structure beyond the given situations, s/he may perform extending.

'Reflection Generalizations' as Ways of Understanding

The ways of understanding of generalizing act are characterized by identifying the product; in other words, the generalization expression. The expressions are categorized as reflection generalizations. At the end of the generalization process, the student can produce a pattern, rule or a definition.

DNR-Based Instruction

DNR-based instruction is an approach that demonstrates what component of mathematics should be taught in schools to improve the success of students and how it should be taught. The answer to the question "What should be the content of mathematics taught in schools?" is hidden in the duality principle. Duality claims that the knowledge of mathematics comprises both ways of thinking and understanding, and ways of understanding produced are influenced by ways of thinking, and vice versa (Harel, 2008b). Because ways of thinking and understanding influence each other, the intention should be to improve both simultaneously. Understanding what is going on in the mind of the student and supporting his/her mental development is key to improving his/her ability to learn. However, it is known that more emphasis is given to the development of ways of understanding that express the product because of the factors such as exam anxiety and increasingly tight schedules for learning content (Baki & Kartal, 2004). The improvement of mathematical knowledge and, in this respect, mathematical thinking, is related to the development of these two types of knowledge.

Improving ways of thinking and understanding is related to the principles of necessity and repeated reasoning. According to Harel (2008b), the only way to learn is to solve problems, as we only experience disequilibrium in the mind when a problem is encountered. Individuals first take the new situation into his/her existing cognitive schema. However, when the new problem situation does not fit the existing cognitive schema, s/he forms a new cognitive schema by arranging the existing schema. Learning is a continuum of disequilibrium–equilibrium phases manifested by (a) intellectual and psychological needs that instigate or result from these phases and (b) ways of understanding or ways of thinking that are utilized and newly constructed during these phases (Harel, 2008b, p. 897). At this point, when the mathematics education dimension is considered, the mental and psychological needs of the student should be met in order for learning to be realized. Motivation, the interest towards learning, desire, and willingness compose the psychological needs. When a person faces a problem, it is necessary that s/he has the desire to solve the problem and show perseverance. On the other hand, the mental needs can be defined as providing the mental confusion state that enables obtaining new knowledge from his/her knowledge (Harel, 2008b, p. 898).

After the intellectual need is formed in students' mind to solve the problem and the desirable ways of understanding and thinking are developed, it is necessary for the student to internalize and organize this information by repeated reasoning principle. Cooper (1991) claims that repeated reasoning is necessary to form rich cognitive networks that are interconnected. Conflict should arise in the mind in order for new cognitive networks to be formed. It is difficult and complex for children to "do mathematics". Therefore, mathematical experiences are of critical importance in order for children to deepen their knowledge and comprehension regarding mathematics as well as their communication with their peers and teachers (Kieren & Pirie, 1991). The problems presented to students should not be routine or repetitive. On the contrary, the problems that students encounter must help students to internalize and reorganize the desirable ways of thinking and understanding they formed, and respond to the students' changing intellectual needs. In other words, problems should be presented in a way that triggers students' thinking.

In order to achieve effective teaching, the teaching process must be learner-centred. Therefore, in the process of forming knowledge, the cognitive functions that take place in the mind of the student are of great importance. A teaching process wherein the mental actions of the student are recognized and considered will be more effective in forming their knowledge. In this regard, determining how students understand and think in terms of their generalization processes gains importance. In this study, the generalization processes of students were investigated from the perspective of students. Therefore, the generalization taxonomy, which is based on the learner-centred approach, was used to analyse the generalization processes. At the end of the study, two new categories were recommended to be added to the taxonomy. This study is considered important both on account of the new categories proposed and on account of the fact that the study is based on what transpired in the students' minds.

According to DNR-based instruction, ways of thinking and understanding should be developed together. For this reason, the two-dimensional structure of knowledge should be taken into consideration. In addition, an instruction that fulfils students' intellectual needs and that provides students with the opportunity to internalize, organize, and modify this knowledge should be emphasized. It is thought that desirable ways of thinking and understanding can be improved in a student's mind, considering these factors. In this study, a DNR-based teaching experiment consisted of some algebra topics was developed and conducted in a classroom of eighth-grade students. Therefore, the goal was to determine the students' ways of thinking and understanding while making generalizations. In this regard, answers to the following sub-problems were sought:

1. What are the eighth-grade students' ways of thinking regarding generalization processes?

2. What are the eighth-grade students' ways of understanding regarding generalization processes?

Methodology

In this part of the study the design of the research, data collection tools, data analysis process were stated in detail.

The teaching experiment methodology (Cobb & Gravemeijer, 2008; Steffe & Thompson, 2000) was employed in this study to follow students' development and to determine ways of thinking and understanding when generalizing. The study consisted of three stages: preparation for teaching in which observations were made, the teaching experiment was designed, and the participants were selected; teaching in which pre-application was conducted, the teaching process was carried out, one-to-one interviews were conducted with students, and a continuous analysis was carried out; and analysis in which the data were studied retrospectively. Data collection process is given below (Figure 1).

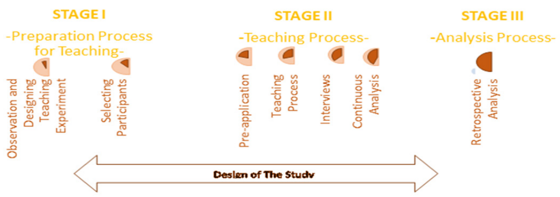

Figure 1. Design of the study

Stage I – Preparation Process for Teaching

During the preparation for teaching stage, classroom observations were made, the teaching experiment was designed, and the participants were selected. In order to determine the difficulties that students encounter in their algebra courses, the researcher followed and performed various teaching practices in seventh- and eighth-grade algebra courses for three terms during the 2014-2015 academic year and the autumn term of the 2015-2016 academic year in the school where the study was conducted. The topics of patterns and relations, the analysis of change, and equations were determined as topics of the teaching experiment based on students' difficulties that the literature points out. The teaching experiment lasted for seven weeks of which one week was allocated for the pre-application and six weeks were allocated for the actual sessions. During the teaching experiment, the students were asked to complete 37 tasks. The task distribution related to the subjects according to the week they were carried out is shown in Table 1.

Table 1. *Distribution of tasks according to the subjects and weeks*

	Subject	Tasks	Day	Mathematical Actions
1st Week	In order for the students to have an idea about how to study and to get accustomed to the camera, questions similar to those presented in the teaching experiment were studied.			
2nd Week	Patterns and Relationship	Task 1, 2	1st Day	Applying different ways to find the rules governing shape patterns
		Task 3, 4, 5	2nd Day	
		Task 6, 7, 8	3rd Day	Determining the rules governing number patterns presented in a table and analyzing the relationships
3rd Week		Task 9, 10	4th Day	Drawing a generalization about geometrical shapes
		Task 11, 12	5th Day	Discovering patterns in number tables
	Analysis of Change	Task 13, 14	6th Day	Investigating the concept of education in terms of analysis of changes
4th Week		Task 15, 16	7th Day	Analyzing relationships and showing them by means of tables and graphs
		Task 17	8th Day	Analyzing the relationships in a table, forming a rule, and switching between multiple demonstrations
		Task 18, 19	9th Day	Recognition of the velocity concept by means of the burning candle question; recognition of the relationship between slope and velocity by means of the robot's journey question
5th Week		Task 20, 21, 22, 23	10th Day	Investigating the changes of velocity concept in terms of analysis Drawing y=mx and y=mx+n lines
	Equaitons	Task 24, 25	11th Day	Recognizing patterns in tables and graphs
		Task 26, 27	12th Day	Interpreting the relationships between non-linear variables Writing equations for the lines presented on given graphs
6th Week		Task 28, 29	13th Day	Showing the relationship between two variables in the form of a curve Writing an equation for the problem given verbally
		Task 30, 31	14th Day	Interpreting linear and non-linear relationships in a table Drawing graphs with different shapes (such as column graph)
		Task 32, 33	15th Day	Showing and explaining the relationship between variables in a table in the form of equations and graphs Analyzing various solutions for problems given verbally and showing the relationships on a graph
7th Week		Task 34, 35	16th Day	Showing and interpreting two lines on a graph, one of which is increasing and the other is decreasing, Writing equations or showing in graph form the relationship between two variables in comparison with one another when a constant is given
		Task 36, 37	17st Day	Writing equations for a problem given verbally, showing the decreasing line for the problem in a graph and interpreting the relationship between two variables given as points in a coordinate system, in terms of comparing them

As shown in Table 1, the teaching experiment lasted 6 weeks, with the first week being used for pre-implementation and the remaining 6 weeks for the main implementation. Among the implemented tasks, 12 focused on patterns and relationships, 15 on analysis of change and 10 on equations.

Selecting Participants

The study was conducted with 9 eighth-grade students. Operational and conceptual algebra tests developed by the researcher were administrated to a total of 167 eight-graders from the participating school which was a public middle school located in Sivas. The operational algebra test featured 10 questions, which were prepared considering seventh- and eighth- grade-level algebra objectives. This test composed of open-ended and non-routine problems. With this test, the aim was to investigate how students solved algebra problems. The conceptual algebra questions implemented in this study were prepared according to Küchemann's (1981) algebraic thinking levels. The purpose of using conceptual algebra in this study was to discern the students' knowledge of algebraic concepts and thereby determine who would participate in the teaching experiment. For student selection, the criteria were that the student must have a certain background in algebra and possess a certain level of algebraic thinking. Therefore, in this study, the classification of letter symbols and the algebra test, which was composed of questions prepared by the researcher in light of the algebraic thinking levels corresponding to the study's aim, were used.

The students' answers to the tests were categorized as high, moderate, and low according to the number of correct answers. The participants were selected from the high and middle categories by their mathematics teacher. The mathematics teacher applied the inclusion criteria of being capable to reason in mathematics problems and of expressing his or her thoughts clearly for the study. Therefore, the study was conducted with a total of 9 students (four males, five females). Pseudonyms were used for the participants throughout the study.

Stage II – The Teaching Process

This stage consisted of pre-application, which aimed to improve students' familiarization with studying in front cameras and informed students about how to study; the teaching process in which the teaching experiment was carried out; one-to-one interviews, which were conducted with two students at the end of each session, the evaluation of the day by the teacher and researcher at the end of each day; and the continuous analysis consisting of the changes of work share if necessary.

The sessions were conducted by applying the following steps: First, each student was given a worksheet describing the tasks and then instructed to study the tasks individually. Following the individual study, group discussion sessions started, where the sharing of ideas by each of the students served to create a group discussion environment. At this step, the researcher and teacher walked among the groups, listened to the discussions, and asked questions to help direct the discussion. After ensuring that all of the students had the opportunity to talk about their ideas, a class discussion was conducted to think about possible methods for reaching solutions. The researcher asked "puzzling" questions during both group discussions and class discussions to explore the students' minds and to help them better express what they thought. In this way, each student's proposed solution was discussed, and the methods of reaching a solution were identified. The implementation of the teaching experiment was carried out first by placing participants into groups of three. The study was conducted three days a week during a six-week period after school in mathematics classroom. Each session lasted between 70 and 100 minutes. Sessions were recorded by four cameras; one camera recorded each group and one recorded the entire classroom.

At the end of each session, two or three questions were given as homework in order to encourage students' repeated reasoning of the related topic. Therefore, students' homework was used as a means of supporting their ways of thinking and understanding as well as providing their repeated reasoning.

Interviews

After each session, one-to-one reviews of the homework of each day were conducted with Ali and Gül. One-to-one interviews were conducted with these students and recorded. Each interview session lasted between 15 and 90 minutes.

Continuous Analysis

The overall data analysis consisted of two stages: continuous analysis and analysis of all data obtained at the end of the study. The continuous analysis was conducted through the researcher's and teacher's evaluation of each stage. The continuous analysis allowed the researchers to investigate whether the implemented teaching experiment of the day facilitated the target change in students' understanding and to make the necessary adaptations for the future application (Molina, Castro & Castro, 2007; Simon, 2000). At the end of each session, the teacher and researcher exchanged ideas, and a brief analysis of camera records and students' worksheets were made in order to arrange the task to be included in the next lesson.

Data Collection Tools

The second stage in which the teaching experiment was implemented also included data collection. The camera records of each session, camera records of group studies containing students' discussions, one-to-one interviews with two students after each session, interviews with each student about their opinions of the teaching experiment process, students' worksheets, students' logs, and the observation notes of the teacher and researcher constituted the data collection tools.

Data Analysis

The camera records were transcribed using the continuous and retrospective analyses. Content analysis was used to analyse the students' worksheets, logs, and the researcher's and teacher's logs. Data analysis process also constituted the third stage of the study.

Retrospective Analysis

The retrospective analysis was the last stage of designing the teaching experiment. It is the process of evaluating the data within a more comprehensive theoretical framework (Cobb, Jackson, & Dunlap, 2014, p. 20). The data obtained in this study were quite comprehensive since the data collection tools were varied and included qualitative data such as video records of various occasions, students' worksheets and logs, the researcher's notes, and teacher's logs (Molina, Castro & Castro, 2007).

The video records obtained were transferred to the Camtasia Studio 9 software. Therefore, it was possible to view the four different camera records of each lesson and to listen at the same time what was spoken in groups while holding a discussion. Powell, Francisco, and Maher (2003) developed a model to analyse video records obtained in order to determine the development in students' mathematical thinking. Being designed to examine the development of mathematical thinking, this model consists of 7 interrelated steps that are nonlinear. The video records obtained within the scope of this study were analysed considering this model.

Content analysis method was used to analyse the data obtained from the students' worksheets, logs, and the researcher's and teacher's logs. NVivo 8 software, a qualitative data analysis program that facilitates coding, grouping, and linking of data (Kuş, 2006), was used in the process of forming themes and categories.

Results

In this part of the study, the findings were arranged in accordance with the sub-problems. According to this, firstly the ways of thinking about the generalization process of the students and the ways of understanding that the students revealed after the generalization processes were determined. The findings were supported by students' expressions and worksheets.

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Students' Ways of Thinking Regarding Generalization Processes

The aim of the tasks regarding patterns and relations was to enable students to develop different ways of thinking to find the rules of patters presented as figural or number; to recognize relations in patterns presented in multiple representations; and to express these relations algebraically and verbally. The aim of the tasks regarding the analysis of variance was to enable students to investigate the variance of variables presented with multiple representations such as algebraic, verbal, table, and graphic formats according to each other, and to demonstrate this variance with multiple representations. The aim of the tasks regarding the equations was to enable students to demonstrate the variances that they analysed with multiple representations such as algebraic, verbal, table, and graphic formats. Approximately 250 generalization actions were coded. Examples from each generalization action were presented in this paper since it would not be reader-friendly to include all generalization actions.

Relating

Relating is achieved when a student creates a relation or establishes a connection between two or more situations, cases, or problems. The actions performed within the category of relating are random rather than intentional. When a student looks for a relation, he or she does not know how this relation is established. Relating is achieved by either connecting situations or objects.

Relating Situations

A student engages in relating situations if the act of relating involves two or more problems or situations. The situation in this context is anything that is perceived by the students as a situation. A student might perform connecting back if he or she relates a new situation with the existing problem, example, or situation. For example, one of the tasks was about finding the rule of a pattern of which the first three steps were given. The students attempted to discover the rule differently from their teacher's method.

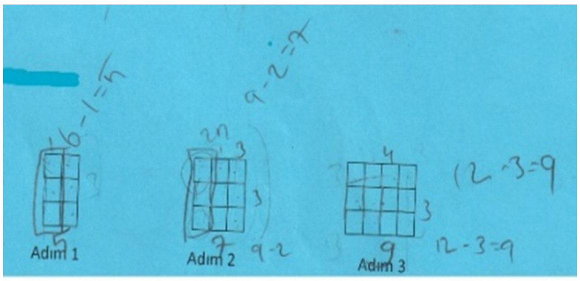

Figure 2. Bartu's worksheet

Bartu: I made all closed figures by adding a square to each. For example, if we add a square in the middle, it becomes a rectangle. The second figure becomes a square by adding two squares in the middle. It becomes a rectangle again if we add three squares to the third figure.

Ezgi: It is like a rectangular number. Then, it goes like a rectangle, a square, a rectangle...

R: Rectangular numbers?

Ezgi: The numbers that we wrote as a rectangle.

Ezgi established an effective relationship between the square numbers and the figures and connected back since she related with her existing knowledge. Despite inaccurately expressing her relation, students' focus is the foremost point. The mathematically inaccurate nature of the situation to which the student relates does not pose an obstacle to the examination of this situation.

A student might develop a new situation that is similar to an existing situation and establish a relation with this new situation. In this case, this student is said to relate with a new situation. When students were asked "to figure out a scenario whose equation can be formulated as y=6" in one of the tasks, Koray answered, "There are 1-million stores. If this would be a 6-million store, the relation between the goods and their price would be y=6. In this case, toys are 6 liras and Cokes are 6 liras". Therefore, the equation was related with a new situation.

Relating Objects

A student might establish relations between mathematical objects on the basis of the similarity of two or more equations, graphs, tables, problems or other objects. The relationship can also be established through assimilating mathematical objects to each other in a visual or formal way.

One of the tasks was about finding the rule of E pattern, the first two steps of which were given, the students connected back with the rule found on the previous day. They attempted to find the area of the figure by adding squares in order to make it a complete figure.

Sezen: There are 15 squares in the first and 28 squares in the second if we complete the area. Later we need to subtract the ones we added.

Melike: There are 15 in the first. We will take out 4... Consider that we should reach this formula (the formula that they found by applying the existing formula).

R: Can you express what you have said as a rule?

Bartu: In the previous lesson, we calculated the area of the rectangle; yet, the short side was a constant. In this case, however, both short and long sides increase.

Sezen: The long side is five in the first figure and seven in the second figure. It increased by two. Therefore, the rule becomes 2n+3.

R: What do you say about the short side?

Gül: 3, 4, 5. It increased by one. Then, the rule is n+2.

R: How many unit squares are there in the rectangle I created?

Ezgi: We calculated the area yesterday. What is multiplied by what?

...

Oğuz: (2n+3) is multiplied by (n+2).

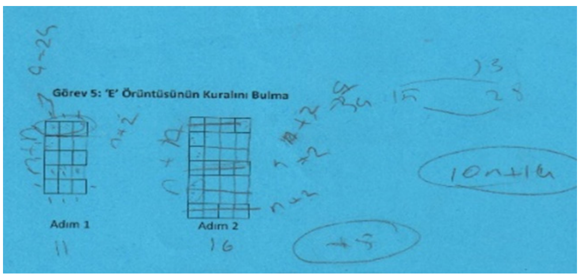

Figure 3. Melike's worksheet

The students established a formal relation between this problem and the problem solved on the previous day; however, they encountered difficulties with the solution.

The solution for the previous problem dealt with the area of a rectangle based on the number of square units it contained. However, the students first initiated a solution by searching two quantities to be multiplied since they had not yet learned the area of a rectangle by relating it with the number of square units included in this rectangle. Later, Oğuz stated that the area could be found by multiplying the short and long sides and, in doing so, related the property.

One of the tasks was about renting DVDs from the most appropriate of three different stores that offer different fare schedules. The students formulated the fee each store charged for one DVD. Bartu designed a table in order to clearly see the three different prices. According to this table, he found the rule regarding the price to be paid to each store and the number of DVDs. Later, he wrote the equations stating the relationship between the number of DVDs, the store, and the corresponding prices.

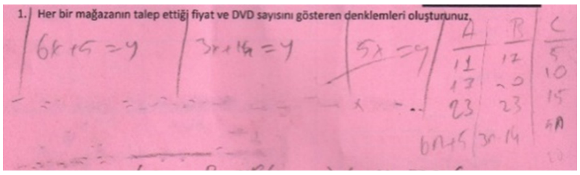

Figure 4. Bartu's worksheet

Bartu examined the table and his equations. Then, he noticed that the rules regarding the relationship in the table and his equations were the same. "The rule we found is the same with the equation we wrote. We wrote n in the rule, and substituted x with n and equalized to y," he said. Bartu realized that finding a rule for the relations in any table and expressing these relations algebraically is the same procedure and shared this with his peers. Therefore, Bartu connected the formal property of the equations with the rules of the patterns he found in the table.

Relating With An Authority

In the task in which the first three steps of the figural pattern were given, and its rule was asked, the students found the rule by transforming the figural pattern into a number pattern. When asked how she found the rule, Melike, like her friends, expressed the rule in the following way:

> *Melike: If we transform them into numbers, they become 4, 7, 10, 13. The difference between them is three; that is to say, 3n. We need to add one to get the first term. The rule becomes 3n+1 if we add one. This holds true for the other figures.*
>
> *R: How did you find this?*
>
> *Melike: Our teacher taught us to do it this way.*

All of the students found the rule of the pattern in this way and explained that "their teachers had taught them in this way". The actor-oriented transfer provides important clues about how an individual relates new situations with the existing cognitive structures in his/her mind (Lobato, 2003). Therefore, the student's focus becomes crucial in this process. In the relating stage of generalization, students relate two or more existing problems and situations based on their properties or formal properties in particular. However, the relations that all of the participants made are different than a problem, rule, or formula that was previously encountered. When finding a rule, students often related it to an authority figure, which, in this context, could be a teacher, textbook, etc. Therefore, this category is included in the stage of relating as relating with authority.

Searching

A student might repeat the operations in order to find similarities. The actions performed at this stage are, therefore, more intentional. Students search for similarities among many examples with the awareness that there is a similarity between them. For example, a student might search for the same relationship based on the examination whether the ratio of the given quantities remains constant. A different student can find the ratio of the given quantities but might not be able to speculate on this ratio. The student who carries out the similar procedure that is taking ratios repeatedly might search for the same procedure. Although searching for the same relationship and procedure are similar to each other, the decision into which category the action carried out falls is left to the researcher. For example, a student might search whether the pattern stays constant in a given task. However, the student cannot think beyond the given situation. Therefore, he or she might search for the same pattern. Students focus on the operations in all three categories. However, students' focus is sometimes the result of the operations, which they carry out to get the same result repeatedly. In such cases, students perform the action of searching for the same solution or result.

During the task in which the first three steps were given and its rule was asked, the students completed the figure to a rectangle by adding squares (Figure 3). Later, they found the rule by subtracting the number of unit squares they added from the total number of unit squares included in the rectangle. Gül and Ali expressed their thoughts and Ali put this rule into words:

> *Gül: If we determine the total number of small squares in these rectangles, and subtract the squares we added from this number, we can find the total number of squares.*
>
> *R: How can we find the number of small squares in the rectangle?*
>
> *...*
>
> *Ali: The side lengths of the unit squares in the rectangle are one unit. Therefore, in the first figure, the long side is three units and the short side is two units. This holds true for the other figures.*
>
> *R: How do we know that?*
>
> *Ali: I remember that our teacher solved a different question in this way. We thought that there were tiny squares in the rectangle and found the area.*

Gül aimed to find the number of squares in the figures by subtracting the number of squares added from the total number of squares included in the completed figure. Therefore, she searched for the same procedure by establishing a relation with the formal property of the rectangle. Ali, on the other hand, stated that the number of squares included in a rectangle can be determined by assuming the size of the square is one unit, like his teacher once demonstrated. Therefore, he expressed that a rectangular area is equal to the number of unit squares included in this rectangle. The rule expressed by Ali, by relating with an authority, is characteristic of the process of searching for the same relationship.

In one of the tasks, students were asked to think on situations that there cannot be talked about slope. The students stated various situations that slope cannot be achieved. Then the researcher raised the following question: "Let us imagine a distance-time graph. The time is constant, does not change; however, the distance increases continuously. For example, imagine that the distance travelled in the third second continuously increases. Would that be possible?"

> *Oğuz: It would not be possible. The distance I travelled in the third second cannot increase by five, six, seven meters because I cannot be in different places at the same time.*

In this task, the students agreed that the distance travelled cannot increase when the time is constant. However, they had difficulties in demonstrating this idea using a graph to state that when there is no change in the x-axis, a change cannot exist on the y-axis. Thereupon, they assumed a constant x point and established a line by combining it with the y value, as established in the previous example. Therefore, the students searched for the same pattern in order to determine whether the pattern applied to the new problem. In doing so, they discovered that the slope could not be achieved since this situation is contextually impossible.

The students considered an alternative approach when working on the task in which the first three steps were given. After working on it, Ali put the "3n+1" rule of the pattern in a different way:

> *Ali:* $\frac{9n+3}{3}$ *works out as well.*
>
> *R: How did you find that?*
>
> *Ali: By trial. Let us substitute 1 for n. Nine times one, plus three equals 12. Divide this by three, and the answer is four. Here is the first term.*
> *Now, I realize that* $\frac{12n+4}{4}$, $\frac{15n+5}{5}$ *can work out, too.*

Ali proposed this rule by relating the number of squares in the steps rather than assuming that multiplying and dividing 3n+1 by the same number would not change anything. Ali's action can be considered under the category of *searching for the same solution or result.*

Searching the same piece

In the task in which the first three steps of the figural pattern were given, and its rule was asked, Sezen found the rule for the pattern in the following way:

Sezen: I have noticed that there are one, two, three, four boxes here (the number of boxes in one row above the steps). There are one, two, three, four boxes here (the number of boxes in the middle row of the steps). There are two, three, four boxes here (the number of boxes in one row below the steps).

R: Can you transform this into a rule?

Gül: The pattern is as follows (for the number of boxes on the above, middle, and below rows in the first step): one, one, two; two, two, three. Then, it becomes n, n, n+1. If we add them up, it is 3n+1.

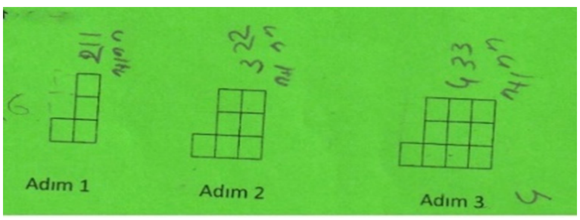

Figure 5. Gül's worksheet

Sezen and Gül broke the pattern down into pieces, related the number of boxes in the pieces with the step number, and derived a rule from this relation. The students' action in this example was different than investigating the relation between the terms composing a pattern. The students' focus here was on the pieces comprising the pattern. Their action was, therefore, to investigate whether these pieces were repeated in all terms and whether this change was constant. This action was thought to be different than the actions which exist in the categories of *searching.* Therefore, this action was added to the taxonomy as the category of *searching the same piece.*

During a different task in which the first three steps were given and its rule was asked, the students developed different ways of thinking. Ali found the rule in a different way and described his approach as follows: "Each step has as many rows and columns as the number of steps. One row, one column in the first step; two rows, two columns in the second step. There are three points in each row and column, which makes six points in total. I multiplied the step number by six. Then, I excluded one point since I counted it twice (the point on the intersection of row and column). I also found the rule for the points I counted twice: 2n-1. I subtracted 2n-1 from 6n. Finally, I found 4n+1".

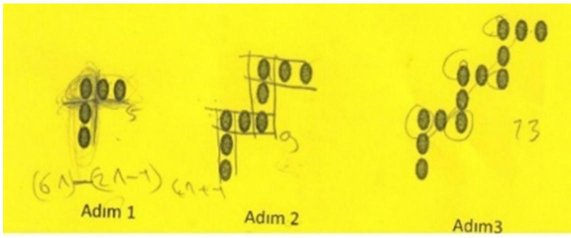

Figure 6. Ali's worksheet

Ali first found the increasing units based on the relation between terms. Subsequently, he broke the term into pieces and found a rule based on the individual units. In other words, he broke the terms into pieces that constitute the term itself, investigated whether these pieces are constant in other terms, and subsequently expressed this rule algebraically. Therefore, Ali's approach was found to be different than investigating the relation between terms and considered under the category of *searching the same piece.*

Extending

A student might perform the action of *extending* if s/he is not only aware of the existence of a similarity relationship but also expresses this relation in a more comprehensive manner that goes beyond the given situation. The extending action leads to a new relation, construction or definition. A student is said to *extend by expanding the range of applicability* if s/he applies the relation to other cases. A student might sometimes *extend by removing particulars* in order to create a situation that is more general than his / her generalizations. The aim of this extension is to identify a general phenomenon that is accurate for every object in a classroom setting. The student can extend and apply the relation to new examples. Therefore, s/he performs *extending by operating*. The student is said to *extend by continuing* an idea or pattern if the pattern is applied repeatedly. What is essential here is to focus on a constant pattern and continue without altering it. The student's focus is on the relation that causes the pattern.

Prior to working on the task regarding the journey of Curiosity to Mars, a short video about the Curiosity Rover was presented to the students. Students were asked to think about the task presented in the lesson and to imagine the scenario. In this task, the change of Curiosity's distance from the world was given by a distance-time table and some questions were asked about the table. The students speculated about the variance of the velocity of Curiosity and attempted to reach a solution by comparing the differences of the values in the table.

> *Elif: The slope is constant if the ratio of the differences among them is constant; otherwise, it is varied.*
>
> *Ali: The slope was eight, nine, and then 10, up to 40 seconds. The velocity was first constant, but then it increased.*
>
> *R: What would you say about the distance travelled in a unit of time?*
>
> *Elif: The distance travelled by a unit of time is eight kilometres up to the fortieth second, and then it increases.*
>
> *Koray: Then, it left a module to accelerate on the fortieth second. Afterwards, it left another piece to accelerate.*

Students first speculated about the slope by comparing the differences in numbers. Afterwards, they *extended by operating* by relating the knowledge that the distance traveled in a unit of time is equal to the velocity. Therefore, they agreed that the velocity is constant up to the fortieth second, and then increases. Koray, on the other hand, by relating with the video he watched, thought that Curiosity accelerated since it left a module and accelerated more by leaving another piece.

During the task regarding Curiosity's journey to Mars, students attempted to demonstrate the information presented in the table in a graph. All of the students created accurately the velocity-time graph. Bartu volunteered to draw the Curiosity's distance-time graph.

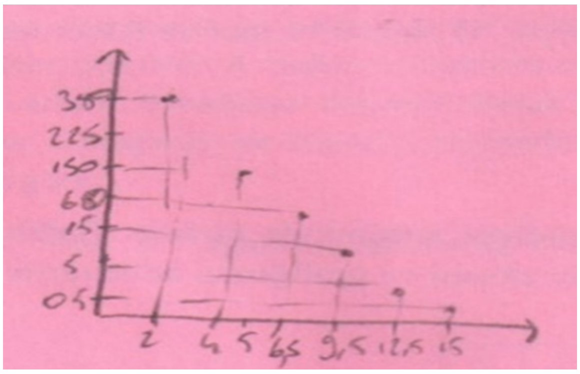

Figure 7. Bartu's worksheet

Regarding this graph, the students made the following comments:

> *...*
>
> *R: What is your independent variable?*
>
> *Oğuz: Time, because it changes independently from others. It also influences the distance.*
>
> *Ali: We solved the problem like we did in the previous lesson. As Curiosity fires from the earth, it moves away from the world. Therefore, the graph would be like this (he draws an increasing line in the air with his hand). However, since it gets closer to Mars, the distance gets even smaller. Therefore, it must be a decreasing graph (line).*

Oğuz determined time as the independent variable. Ali made a connection back to the graph drawn in the previous question. Ali stated that this must be a decreasing graph since the distance to Mars gets smaller, considering that since the distance to earth increases, it must be an increasing graph. Therefore, he *extended by continuing.* Koray, on the other hand, pointed out that the points demonstrated by Bartu should be drawn as a line. He justified his argument as follows: "In the previous lesson, if the line holds true for all the points between, a linear line should be drawn. The distance here always decreases. Therefore, we need to combine the points." Koray established a relation by making this connection with the previous lesson. He *extended by removing particulars* by expressing for each x on the line, the corresponding y values also provide the line.

> *Ezgi: We expressed this as a line. Its slope becomes constant at every point, 350/15. Therefore, the velocity is also 350/15. However, we found the velocity different at every interval; a bit increased, a bit decreased.*
>
> *Bartu: I also noticed that, but I could not decide whether to decrease by the same amount, to decrease a bit more, or to decrease more from which one.*

Ezgi found the slope of the linear line and related it to the Curiosity's velocity. She expressed that the line will be broken at the points where the velocity of the robot changes since Curiosity's velocity is different at different time intervals. At the time intervals where the Curiosity's velocity is high, the line becomes more perpendicular; at the time intervals where its velocity is low, the line becomes more sloping. Therefore, she *extended the range of applicability.*

Students' Ways of Understanding Regarding Generalization Processes

Any kind of product that is revealed as a result of individuals' mental actions is described as a way of understanding of those mental actions. In this regard, a solution to a problem or a proof constitutes ways of understanding of the mental acts of problem-solving or proving. Reflection generalizations indicating the products generated as a result of generalization actions are divided into various categories according to the product that the student reveals. According to reflection generalizations, the written and verbal generalization types expressed by students are investigated within the categories of identification or statement, definition, and influence. Reflection generalizations are quite related to the generalization actions.

Identification or Statement. Students can reveal generalization products by identifying similarities between expressions of situations, properties, general rules or patterns, strategies or procedures, and global rules.

Identifying the Continuing Phenomenon

Students can provide explanations for a continuing property that occurs in their generalization expressions. For example, in the task about the graph demonstration of the possible side lengths of a rectangle whose circumference is 18 units, the students first determined the possible side lengths and presented them in a graph. They combined these points along a line, indicating that each x and y value prove this equation. If the expression used in the generalizations indicates a dynamic relation, a generalization of identifying a continuing phenomenon is said to be performed.

Identifying Sameness

Students can express similarities between two situations, problems, and objects as a product of generalization. The problem or situation that is expressed as similar in this context is done so from students' points of view. The problem or situation that is the students' focus can emerge as the similar property, objects/representations, or the expression for the similarity between the situations.

Sameness of Common property. The possibilities were pointed out regarding the following task: "The distance travelled at certain time intervals by Melike who travelled equally long distances in the same time intervals as Feyza, where Feyza travelled 16 meters in 10 seconds". When the students were asked how they found the answer, they replied that all numbers were obtained by multiplying and dividing by 10 and 16 with the same number. As seen in these examples, the students reached a generalization by expressing the common property between two situations.

Sameness of objects or representations. Regarding the rules of patterns provided during the first days of the teaching experiment, Ali, Bartu, and Enes expressed that the pattern's rule, which is 3n+1, can also be written as '(9n+3)/3, (12n+4)/4, (15n+5)/5, and (18n+6)/6'. When they are asked why they provided alternatives, Enes stated that these also indicate the same rule since dividing or multiplying 3n+1 by the same number does not make a difference.

Sameness of situation. During the task that involved a rectangle with a circumference of 18 units, and its possible side lengths unknown, regarding the slope of [AB], Ali constructed ABD right triangle where [AB] is the hypotenuse. Then, he defined the slope as the ratio of adjacent side to the opposite side since the tangent of the ABD angle gives the slope. Elif answered this question incorrectly. When she was asked about the slope of [BC] by the researcher, she constructed a right triangle where [BC] is the hypotenuse with a thought that these situations are similar. Therefore, she was able to answer the questions correctly.

Identifying a General Principle

Students can express general rules, patterns, strategies, or principles in these situations by utilizing their generalizations. Generalizations obtained in this way are the most widely accepted by mathematicians when they are also expressed in algebraic form. General rule, pattern, strategy, and global rule in this context are all considered under the category of general principle.

General pattern. On the first day of the teaching experiment, Oğuz's reasoned the following: "If we take one square constant, other columns increases in every term. There are triple blocks as many as the number of steps in every term". His argument here regarding the pattern rule given in the first task can be considered as an explanation of a situation increasing among terms.

General rule. Regarding the possible side lengths of a rectangle whose circumference is 18 units, the students expressed the number pairs as (1,8), (2,7), (3,6), and (4,5). More generally, Ali found a rule indicating that x +y=9 if 2x+2y=18, where x is one side, the other side is y, and expressed this rule as "the integers whose sum is 9".
General strategy or procedure: The possibilities were pointed out regarding the following: "The distance travelled at certain time intervals by Melike who travelled equally long distances in the same time intervals as Feyza, where Feyza traveled 16 meters in 10 seconds." When they were asked how they determined these numbers, Koray explained his reasoning as follows: "Since she travelled 16 meters in 10 seconds, dividing them by two, it makes eight meters in five seconds. I can find the distance travelled by Melike at a time by multiplying or dividing the (5, 8) number pair with any number". Koray first determined the smallest number pair. Then, he expressed that he can reach the asked number pair by dividing and multiplying this smallest number pair with any number. Therefore, Koray applied a more general strategy independent of any object.

Global rule. One of the tasks was about finding a rule of some polygon's (triangle, square, pentagon,...,n-side) number of sides, number of diagonals, sum of interior angles. Ali found a rule for finding the number of diagonals of the polygon's and gave the following explanation: "A pentagon has five corners. A corner itself can neither create a diagonal itself nor can it create a diagonal with its two adjacent sides. Therefore, we can draw two diagonals from one corner, which makes 10 diagonals in total from five corners. Since we count all of them twice, we divide them by two. Therefore, it makes five diagonals". Ali, in fact, explained, by reasoning, how to find the number of diagonals using the combination.

Definition

Students might make definitions by expressing the basic characteristics of a pattern, object class or relation on which they work as a result of their generalization actions. Even though it seems unlikely for primary and elementary school students to reach a definition thinking an object class, the definitions mentioned in this context might not necessarily be mathematically accurate or complete. The expression of the basic characteristics of the pattern, class, and relations to the extent noticed by students is considered in this category. In the task where some polygons were given and the rules of numbers of sides, diagonals of polygons were investigated, Melike asked her friends what a diagonal was. In response, Bartu offered the following definition: "From one corner to the other". Ali stated a more accurate definition: "A line segment from one corner to a corner to which it is not connected".

The students investigated the variance of time and distance in the table indicating the change of Feyza's distance to home with respect to time and stated that the ratio of these differences is constant. The students noticed that the distance travelled in a unit of time equals the distance travelled in one second by relating the ratio of differences to the slope. The students stated that if someone travels an equal distance in every second, s/he walks with a constant velocity. The students both noticed the relation between the slope and velocity and provided a definition of velocity, which is the distance travelled in a unit of time.

Influence

A student might use a generalization that was previously obtained as a result of a generalization action in problem situations encountered for the first time. This situation can be confronted in two different ways: either the student makes a new generalization using the previous generalization, or the student makes a new generalization altogether.

The influence of prior ideas. This notion refers to generalization by using prior knowledge in new problem situations. If a researcher does not have much information about the mathematical history of students, the identification of prior ideas can be difficult. In the session in which the slope was discussed, the students talked about various meanings of the slope. Volkan, on the other hand, defined the slope as "tangent 30 or 60." When he was asked about the reason, he said that they used a similar approach in previous problems. He applied his prior knowledge, that tangent 30 or 60 gives the slope, to this context. Therefore, he expressed his idea under the influence of his prior knowledge.

Modified ideas or strategy. Students tend to adapt their existing knowledge as a result of making generalizations to accommodate a new problem situation. Therefore, they modify their existing knowledge. As mentioned previously, Volkan answered the question with his prior knowledge that tangent 30 or 60 give the slope. In the rest of the discussion, various comments on the slope were made and explained. Regarding the analytic geometry dimension, the teacher drew a line on the board and asked for its slope. The students defined the slope as "opposite divided by adjacent", using their existing knowledge. The students were then encouraged to think about this existing knowledge with relation to the angle the line makes with the x-axis. They finally reached the conclusion that the tangent value of the angle gives the slope. Therefore, they developed new knowledge by modifying their previous understanding of how to calculate the slope.

Conclusion

In this study, the thinking and understanding ways of eighth-grade students were determined. In evaluating the teaching experiment process through a holistic view, the following conclusions were reached: The first five tasks in the teaching experiment were on finding the rule governing given patterns. The first days of the experiment involved the students converting shape patterns into number patterns, followed by finding the rule governing the pattern by means of relating it to the rule that their teachers taught. The process of coming up with this pattern result generalization (Harel, 2008a) serves to demonstrate a way of thinking, as the students' focus is on the result of the actions they have taken. However, by the time the students were at the end of the subjects about patterns and relations, they had discovered the rules governing the patterns by thinking in different ways and were observed to be successful in interpreting the relations between the terms. Finding the rules governing patterns from the relationship between the terms is considered as process pattern generalization (Harel, 2008a). This is because the students focus switches from finding the rule according to a formula to finding the rule by means of interpreting the relations. Study on the subjects of patterns and relations lasted 2 weeks, after which the students were able to recognize the relations by thinking in different ways and grew to enjoy the process.

The tasks in which change are analysed through interpretation of the relations are quite important in the process of generalization (Booker, 2009). The sixth day of the teaching experiment started out with a study of the slopes of certain lines according to the analysis of change. During the teaching experiment, the thinking process involving the idea that the slope should be considered in the analysis of changes functioned as one of the most important knowledge acquisitions to be formed by the students. In the first tasks related to this subject, it was observed that the students thought that in the number tables given, there was a relationship between x and y columns only. But after a few tasks, they recognized that, in addition to the relationship between the x and y slopes, intra-relationships on the x slope and the y slope also existed. The students were then able to make comments based on these newly discovered relationships. This development can be considered as a transition from product-oriented thinking to process-oriented thinking. Furthermore, another relationship that the students eventually discovered and were able to recognize on the subject of analysis of change was the one existing between velocity and slope. Here it is important to point out that once they identified this relationship between velocity and slope in the analysis of change, they were able to reach the definition and formula of velocity. On the eleventh day of the teaching experiment, the analysis of change tasks were completed, with the results showing that the students had great success in recognizing the relationships and in interpreting whether changes remained stable or not.

In the tasks on equations, the aim was that the students be able to show the changes they analysed in multiple representations, such as algebraic, table, and graph form. The students had been accustomed to expressing these relations either verbally or in tables in their previous tasks. As one of the key knowledge targets of this study, the students were expected to be able to recognize that a rule (e.g. 3n+1) they had discovered to govern a pattern showed a relationships between two patterns, and from this, they were expected to understand that they could write this relationship between variables in an equation. By the end of the teaching experiment, they recognized that the pattern rule they found stemmed from an analysis of changes in the variables in terms of their initial values, and that any of the equations they encountered thereafter involving these variables did not invalidate the rule they found at the beginning. The students considered this idea to be very interesting and qualified it as the "beauty of mathematics". However, with that said, they experienced certain difficulties in trying to show these relations on a graph. Yet, at the end of the teaching experiment, it was observed that the students had overcome the difficulties they experienced in showing the relations on a graph. Similarly, Elia and Spyrou (2006) reported that in their study, the students had high levels of success in algebraic expression of function but difficulties in expressing this on a graph.

Ellis (2007), in his study on generalization taxonomy, recommended that this taxonomy be used in different subjects and on different samples to observe whether the taxonomy works well with different subjects and patterns. Therefore, in this study, the sample, the teaching experiment, and the algebra subject were changed, and based on an investigation of the answers given by the students, it is recommended that two additional categories need to be added to the taxonomy.

As a result of the teaching experiment, students' ways of thinking regarding the generalization processes were found to consist of relating, searching, extending, and the ways of thinking determined in these categories. The generalization process begins with the recognition of a similarity between the given case, situation, and problem. Associating the similarity between 'what' and 'what' is investigated in the category of relating. This similarity is established from the student's point of view. Since students are at the centre, their focus becomes crucial. On the first day of the teaching experiment, when the researcher asked the students how they solved problems and found rules of patterns, they replied that their teacher had taught them this way. The students related the problem they encountered with a rule taught by an authority rather than relating two problems or mathematical objects. Those who are based on the opinions of an authority rely on the explanations and experiences of this authority. Knowledge obtained in this way is assumed to be true without questioning its why and how (Gambrill, 1999). The teacher or the book as the authority in this context is an external source that is more knowledgeable. Therefore, "relating with an authority" was thought to be included in the category of relating.

When the students were asked how they found a pattern's rule during the first days of the teaching experiment, they simply stated that their teachers had taught them in this manner and were unable to put forward any logical explanation. Therefore, this knowledge can be considered as operational knowledge. The students who found patterns in the way that their teachers had taught them were not able to understand what it means 'to find the rule in a different way' when they were asked to do so. During the first two weeks of the teaching experiment, the students defended their knowledge using an authority figure. From then onwards, their knowledge shifted from being operational to contextual since they began questioning and thinking from different point of views.

In conjunction with relating and expressing similar situations, whether similarity/change is constant is investigated. This is referred to as searching in the generalization process. On the second day of the teaching experiment, Ali attempted to find a rule regarding the pattern by breaking the terms into pieces: "Each step has as many rows and columns as the number of steps. One row, one column in the first step; two rows, two lines in the second step. There are three points in each row and column, which makes six points in total. I multiplied the step number by six. Then, I excluded one point since I counted it twice (the point on the intersection of row and column). I also found the rule for the points I counted twice. It is 2n-1. I subtracted 2n-1 from 6n. Finally, I found 4n+1." If Ali had searched whether a relationship between the steps stays constant (for example, we added four to the first step and got the second step; we added four to the second step and get the third step...), these actions would be considered under the category of searching for the same relationship. However, Ali first determined the pieces that constitute the first step and then investigated whether the second and third steps are composed of these pieces too. Since it was investigated that whether the pieces are constant in other steps; that is to say, the focus is on the pieces, a new category was decided to be added to the taxonomy. The strategy Ali used to find this rule is defined by Rivera (2010) as "deconstruction generalization". Since pattern formation is a personal and constructive action, students should coordinate their comprehension and symbolic knowledge in order to make inferences about known and unknown steps of a pattern (Rivera, 2010, p. 298). In one study, it was reported that students divided a given shape into better known components and obtained a rule from the newly formed shape (Chua & Hoyles, 2011). Since this study investigates students' generalization processes based on the generalization taxonomy, it becomes necessary to adapt this strategy to the nature of the study. Therefore, it is appropriate to add this action performed by Ali and other students to the category of searching as "searching for the same piece".

Determining whether the relationship is constant is followed by the process of extending this relationship in a mathematically appropriate and accurate way. The expression of the determined relationship in a general form beyond the given situation is considered under the category of extending. The students' ways of thinking were determined through an analysis of the data obtained from the teaching experiment and organized as follows by adapting from the study conducted by Ellis (2007) and adding new categories as a result of the present study:

Figure 8. Students' ways of thinking regarding generalization process

As seen in the figure, students' ways of thinking are determined in the categories of relating objects and relating with an authority, searching the same piece, relation, pattern, procedure, solution/result; extending the range of applicability, extending by removing contextual details, operating, and by continuing.

In this study, students' ways of thinking about generalization were analysed according to the generalization taxonomy, on the basis of actor-oriented transfer. A review of the literature showed that the ways of thinking regarding generalization have been investigated based on different approaches. Barbosa (2011), for example, determined the strategies used by a group of 54 sixth-grade students by asking two questions requiring near and far generalizations to be made. Based on the results of the study, the way of thinking applied to generalization action was deemed to be recursive thinking, which means that the students solved the problems by using counting, whole-object, difference, explicit, guess and check strategies. When the data obtained in the present study is looked at from this perspective, it can be observed that Melike, for example, in the question about finding the E pattern's rule (Figure 3), used recursive reasoning, as she counted the number of squares in the first and second step and found the pattern rule by subtracting. In this way, she turned the shape pattern into a number pattern and found the solution by using the difference strategy. Recursive reasoning is quite widespread among students. Studies can be found in which students applied similar strategies by using recursive reasoning (Barbosa, 2011; Lannin, 2003; Orton & Orton, 1999; Stacey, 1989; Tanışlı &Yavuzsoy Köse, 2011; Zazkis & Lildejahl, 2002). In a study by Stacey (1989), a pattern question on finding the correct shape that followed another shape in order to discover the rule was presented to the students, and to solve it, they used a counting strategy, which was expressed in this case as "the number of lights increased by 4 in each shape". Moreover, in the same study, there was a question where students were to find the difference between terms; they found the solution based on the fact that there was a common difference for each term.

The practice of finding the rule governing a pattern by turning a shape pattern into a number pattern when a shape pattern is encountered is quite widespread. In the first days of the teaching experiment, it was observed that the students conducted their transactions by turning shape patterns into number patterns. Becker and Rivera (2006), in their study, reported that those who preferred a numerical approach in the generalization process conducted numerical transactions. The numerical approach can be described as turning a given shape pattern into a number pattern and using this number pattern to determine the rule (Becker & Rivera, 2005; Tanışlı & Yavuzsoy Köse, 2011). There are strategies for finding the solutions to number patterns. These strategies included i) comparing the terms of a given number sequence with another similar number sequence whose rules are already known; ii) putting each term in the number sequence into the place of the preceding term; iii) finding a formula by applying the differences method (Chua & Hoyles, 2011).

In the present study, on the question involving a shape pattern whose first three steps were given and a rule was to be applied (Figure 5), Gül first determined the number of squares in the given steps. Based on the increase in the squares in each step, Gül found a rule governing the number of squares to be found in the following steps, and therefore, she conducted visual thinking. Visual thinking is described as expressing verbally the way to complete a sequence, even if the shapes are not seen. Students who conduct visual thinking focus on the structural characteristics of the shape (Becker & Rivera, 2005, 2006). In questions requiring that the structural characteristics of the shape be considered, the focus can easily switch to the question of “how many?”. Friel and Markworth, (2009) in their study, presented a pattern to students and asked the near and far steps of this pattern. For instance, when the smiley number in the 43rd step was asked, the students responded that in each of every three branches there were 43 smileys, while in the section where these branches merged there was an extra smiley. Students who employ visual thinking try to explain the relationship between shape and number of steps on the basis of the constant relationship in given shapes (Becker & Rivera, 2006).

The products that the students revealed as a result of their generalization actions are categorized as reflecting generalizations. The students identified a continuing phenomenon, the sameness of the situations, or a global rule. Further, they identified similar properties between two or more situations, objects, or representations. Nearly all of the tasks in the implemented teaching experiment contain at least one representation of verbal, algebra, table, or graph; in other words, the same situation is expressed in different ways. This is described as the expression of a similar object, or representation. Students might define a general rule between the relations as the product of generalization, a general pattern rule by noticing the pattern among relations, or general strategies/procedures regarding the solution. They can determine and express a mathematical rule as a general principle and define a set of objects as the product of generalization, even though they might not necessarily be mathematically accurate or complete rules. What is defined, however, is produced by the student’s own point of view. In other words, the student might reveal a generalization product under the influence of his or her experience. If this is incorrect, the product revealed under the influence of prior knowledge is changed and modified. As a result, the ultimate product is a form of modified prior knowledge.

Recommendations

Students learn by relating new knowledge with existing knowledge. The more effective this relation is, the more learning is achieved. Therefore, topics should be taught by relating new knowledge with students’ prior knowledge. Relating is the first step of generalization process, which students begin by searching for similarities between problem situations, objects, and other phenomena. Students should be encouraged to think in a way that allows these similarities to surface. Therefore, they should be equipped with the proper tools to do so. They should be provided to think about, question, and obtain knowledge they encountered in not only mathematics lessons but also daily life. Therefore, it should be ensured that they acquire the habit of thinking about why it is so. The people who can think and question, required by this age can only be raised in this way. Ellis (2007) developed the generalization taxonomy with eight-grade students on the topic of linear functions. This study was conducted with eight-grade students on the topics of pattern and relations, the analysis of variance, and equations. Through the analysis of different topics with the generalization taxonomy, new categories were added to the taxonomy. Further studies can be conducted with different participants on a different topic in order to contribute to the taxonomy.

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