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Relational thinking: The bridge between arithmetic and algebra

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Abstract

The purpose of this study is to investigate the development of relational thinking skill, which is an important component of the transition from arithmetic to algebra, of 5th grade students. In the study, the qualitative research method of teaching experiment was used. The research data were collected from six secondary school 5th grade students by means of clinical interviews and teaching episodes. For observing the development of relational thinking, pre and post clinical interviews were also conducted before and after the eight-session teaching experiment. Qualitative analysis of the research data revealed that the relational thinking skills of all the students developed. It was also found that there was an interaction between the development of fundamental arithmetic concepts and relational thinking; that the students developed concepts related to arithmetical operations such as addend and sum; minuend, subtrahend and difference; multiplicator and product; and dividend, divisor and quotient. Moreover, students were able to use these concepts effectively although they failed to provide formal explanations about the relations between them. In addition, the students perceived the equal sign not only finding a result but also as a symbol used to establish a relation between operations and expressions.

Keywords: Mathematics teaching, Secondary school students, Relational thinking, Equal sign, Teaching experiment

Introduction

Mathematics is among the most important tool for the development of thinking skills that individuals need to solve their daily life problems. On the other hand, mathematics is considered to be one of the most difficult subject areas. One basic reason what students find mathematics difficult to learn is that it consists of a specific network of abstract relations, and algebra is the area which includes these abstract relations most.

In school mathematics, algebra refers to use of symbols to state and manipulate generalizations within the context of numbers (Lee, 1996). According to this definition, in school mathematics, algebra has an image prioritizing the solution of equalities and inequalities via symbolic manipulations (Watanabe, 2008). Algebra teaching requires using the language of algebra meaningfully, examining the relations between quantities and supporting the generalization process, and it is problematic in terms of helping students give meaning to related concepts (Blanton, 2008; Carpenter, Levi, Franke & Zeringue, 2005; Dede & Argün, 2003; Lee, 1996; Usiskin, 1997). Even though it is first taught in secondary school years, importance should be given to the development of skills and concepts that will facilitate transition to algebra via student experiences and in-class discussions in early stages. In addition, the number of studies putting emphasis on this importance and defending the need for teaching algebraic thinking at early ages is increasing day by day (Akkan, Baki & Çakıroğlu, 2011; Carpenter & Levi, 2000; Carpenter et al., 2005; Kieran, 2004; Warren, 2009).

In related literature, there is no consensus on what algebraic thinking at an early age is or on what components it may have (Cai & Moyer, 2008). Moreover, it could be stated that algebraic thinking is necessary for the analysis of deeper mathematical structures rather than arithmetic and procedural fluency. Boulton et al. (2000) put forward a three-phase model for the development of algebraic knowledge to examine the developmental steps of algebraic thinking. According to this model, whose phases are called “arithmetic”, “early algebra” and “algebra”, it is claimed that algebraic thinking in students is developed consecutively and that transition to a higher step will not be healthy if the current step has not been developed efficiently. In arithmetic, as the first step of this model, students are expected to know the fundamental properties of operations such as commutative, associative and distributive, to do work backward and to be aware of the equal sign. Therefore, it is obvious that arithmetic constitutes the basis of algebra teaching despite the differences resulting from the natures of arithmetic and algebra (Herscovics & Linchevski, 1994; Kieran, 1981; Knuth, Alibali, McNeil, Weinmberg & Stephens, 2005; Knuth, Stephens, McNeil & Alibali, 2006). It is also seen that students make use of

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their experiences in arithmetic in transition to algebra (Hersovics & Linchevski, 1994; Mcneil & Alibali, 2005). The deficiencies resulting from the way of learning arithmetic during transition from arithmetic to algebra may also have influence on the development of algebraic thinking. For instance, learning subtraction without focusing on the relations between minuend, subtrahend and difference might lead to deficiencies resulting from the way of teaching arithmetic. This situation causes elementary school students to perceive arithmetic as a set of rules. Instead of memorizing the rules directly, it is a necessity for students to see the relations underlying the rules and to develop fundamental arithmetic skills (Knuth, Stephens, Blanton & Gardiner, 2016). Development of fundamental arithmetic skills allows writing down number sentences with mathematical symbols, understanding the fundamental features of operations and conceptualizing a number in a wide variety of forms (5=7-2, 5=3+2, etc.). Students can solve number sentences by focusing on the relation between numbers (Molina & Ambrose, 2006). This focus requires relational thinking, which has an important place in the development of algebraic thinking.

Relational thinking mostly concerns examining the relations between the given quantities rather than finding the result of operations. To clarify, relational thinking involves use of fundamental properties of numbers and operations for the transformation of mathematical sentences. Koehler (2004) points out that relational thinking provides a different perspective for arithmetic and plays a key role in teaching/learning it. This key role brings about two benefits which allow students not only to restructure arithmetic operations to change the given calculation but also to transform the number sentences with the use of fundamental arithmetic properties (Koehler, 2004). In relational thinking, the mere purpose is to help students become aware of the fact that both sides of equation represent the same numbers without doing any calculations. Therefore, for relational thinking, first, students should use the relational meaning of the equal sign (Boulton et al., 2000; Carpenter & Franke, 2001; Yaman, Toluk & Olkun, 2003). On the other hand, a number of previous studies conducted after 1980s demonstrated that most students have serious misconceptions regarding the meaning of the equal sign (Behr, Erlwanger & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Saenz-Ludlow & Walgamuth, 1998), and recent studies (Li, Ding, Capraro, & Capraro, 2008; Matthews, Rittle-Johnson, McEldoon &Taylor, 2012; McNeil & Alibali, 2005) support those previous results as well. As the key to relational thinking, students are supposed to understand that the equal sign refers to the relation and balance between numbers (Carpenter, Franke & Levi, 2003) not to a direction (Kieran, 1981) or the result of an operation. True/false and open number sentences can be used as an important tool which will allow students to start thinking about relations, to learn how to represent these relations and to express the meanings they formed, or which will help develop relational thinking and learn the relational meaning of the equal sign (Carpenter et al., 2003). For instance, in the open number sentence of "28+35=29+ ", it is possible to find the number to be written down in the blank via relational thinking without doing any calculation. When the quantities in both sides of the equation are examined, it is seen that one of the quantities (28) increases to 29. Therefore, it could be concluded that the other number (35) must decrease to 34 in order for the sum to be equal. For this reason, by focusing only on the relations between quantities, the number to be written down in the blank can be found. The answer to this sample question can be found by establishing the relation between numbers, while some number sentences can be dealt with based on properties of operations. To illustrate, students can transform the sentence of "(85+69)+15" into "(85+15)+69" using the associative property, while they can transform the operation of "9x7" into "(10x7)-7" by combining the tens and ones. In this way, they will use the fundamental properties of operations to facilitate their calculations. The most difficult and striking one for internalizing the properties of operations is the distributive property. Distributive property is essential for understanding the multiplication and for developing multiplicative reasoning. Carpenter et al. (2005) state that when students learn the relations involving the distributive property via a method of teaching based on relational thinking, they can recognize multiplicative relations more easily and produce more effective strategies for the multiplication of multi-digit numbers. For instance, in the open number sentence of "8+4=4x(_+_)", students first find the number of 12 by doing the operation of 8+4 and then find the result of the operation as 3 by dividing the number of 12 by 4. On the other hand, a student with the capability of relational thinking can find the numbers to be written down in the blanks by recognizing the fact that the numbers of 8 and 4 are two-fold and one-fold of 4, respectively. Here, the student first recognizes that the common multiple is 4 and then sees that there are three folds of 4 on the left side of the equation. As a result, the student thinks there must be three folds on the right side as well. Therefore, he or she can place the numbers of 1 and 2 considering the commutative property. On the other hand, it may not be so easy to see the relations between the numbers required by the distributive property. Thus, this is not a spontaneous process that students automatically go through. In this respect, it is a must for elementary and secondary school students to take education supporting relational thinking so that they can internalize properties of operations, examine how to manipulate numbers and explain the changes and relations.

In literature, most researchers focused on the meaning of the equal sign and equivalence concept. Students' operational or relational understanding of the equal sign has influence on relational thinking and on understanding the algebraic concepts they will learn in the future. It is seen that most of these studies were carried out with elementary school students from the first three class grades (Carpenter et al., 2005; Koehler, 2004; Molina & Ambrose, 2006; Molina, Castro & Mason, 2008; Molina & Mason, 2009) that some focused on the meaning of equal sign (Baroody & Ginsburg, 1983; Falkner et al., 1999; Kieran, 1981; Matthews et al., 2012; McNeil & Alibali, 2005; Warren, 2006; Yaman et al., 2003) and that some of them investigated use of the equal sign in math course books and the extent to which these books supported relational thinking (Köse &Tanışlı, 2011; Li et al., 2008; McNeil et al., 2006). In one study conducted to systematically evaluate second grade to sixth grade students' knowledge of mathematical equivalence (Rittle-Johnson, Matthews, Taylor & McEldoon, 2011), the researchers found that students give meaning to the equal sign in a constant progress, from operational to relational. The study demonstrated that especially most of the fifth grade students took the operations in both sides of the equation and that they were aware of the relational meaning of the equal sign besides its operational meaning. This result shows that the fifth-grade class has a vital place in students' giving meaning to the equal sign. In the light of these studies, the present study, which aimed at determining and developing the relational thinking skills of students, is thought to contribute to the related literature. In this respect, the present study tried to investigate the changes in in secondary school fifth grade students' relational thinking skills before and after the teaching process based on relational thinking.

Focusing on the change in the students' relational thinking skills, the present study differs from others which examined secondary school students' misconceptions and their thinking strategies they applied to solve open number sentences (Hunter, 2007; Stephens & Ribeiro, 2012) as well as from other studies which investigated students' effective thinking methods for dealing with the number sentences including addition-subtraction (Stephens, 2006). In literature, there is one study in which a teaching model was designed to determine and develop secondary school students' relational thinking skills (Napaphun, 2012); on the other hand, the present study included a wider variety of number sentences and those combining the associative and distributive properties and focused on the students' understanding of these number sentences in the teaching process and on their use of these operational properties while dealing with the operations. In line with this research question, first, the students' relational thinking skills were determined. Following this, the teaching process was designed to develop these skills. Lastly, the changes in the students were examined via the interviews held with them.

Method

In this study, which involved the planning, application and evaluation of a teaching process for the development of students' relational thinking skills, the method of teaching experiment was applied.

Participants

The participants in the study included six secondary school 5th grade students attending a public school from a moderate socio-economic level in the city of Eskişehir. While selecting the participants, the criterion sampling method, one of purposeful sampling methods, was used. Accordingly, three basic criteria were taken into consideration. The first criterion included selection of the 5th grade students whom one of the researchers had taught mathematics and thus had the opportunity to know well and observe. The second criterion was selecting students with higher spoken skills and from different levels of academic achievement. The reason for considering different achievement levels was to examine the relationship between relational thinking and achievement levels and to reveal different/various reasonings of students. For this criterion, views of two teachers (both the researcher and the 4th grade teacher) were taken into account. Lastly, participation to the study was on voluntary basis, and the necessary consents of the students, their parents and the Ministry of National Education were taken. While presenting the findings, the students' names were kept confidential, and nicknames were used.

Data Collection

In the study, the research data were collected via clinical interviews, instructional videos, diaries and worksheets. However, the clinical interviews constituted the basic data source of the study. Instructional videos, diaries and worksheets were used to shape and revise the teaching process.

Clinical Interviews. The clinical interviews were held in two phases: before and after the teaching process. For the preparation of the questions to be directed during the clinical interviews, studies conducted by Koehler (2004) and Carpenter et al. (2003) were taken as basis. In the light of these studies, the questions were prepared considering three headings: "Asking the meaning of the equal sign", "Open number sentences" and "Evaluating whether the given number sentences are true or false**"**. Following this, field experts were asked for their views about the questions prepared. Depending on their views, the number of the questions was decreased, and a pilot application was conducted with a group of individuals with characteristics similar to those of the research participants. The interview questions arranged in line with their views were piloted with a group of individuals similar to the participants of the study. In the pilot study, which lasted two days, certain difficulties were observed: the students experienced time-related problems, and the questions were in an order starting with easier questions and ending with harder ones. Therefore, the number of the questions was decreased, and the clinical interview questions were re-arranged. The questions which were changed were piloted again with three other students from a different class in the same school, and the clinical interview questions were finalized as in Table 1. A total of 32 number sentences with sub-questions were given to the students. The questions directed in the last interviews were different versions of the number sentences given during the preliminary interviews.

During the clinical interviews, such questions as "Would you find the answer without doing any calculation", "Why do you think it is correct?", "Could you please explain it again?", "If you wish, you may think about it more as we still have time", "Well, how so fast did you find the answer?", "Why did you think that number is the right one?" and "Are other different numbers possible?" were directed to the students (Clement, 2000) for the purpose of defining equality, determining relational thinking skills and observing the development of these skills. The preliminary interviews lasted minimum 55 minutes and maximum 93 minutes, and the last interviews lasted minimum 26 minutes and maximum 57 minutes.

Teaching Process. Teachers, who act as a guide for in-class discussions, have big role in helping students give meaning to the equal sign and interpreting the equal sign as well as in their relational thinking (Falkner et al., 1999). Thanks to the in-class atmosphere created by the teacher, a discussion environment can be established, which will allow students to make related generalizations. In order to establish this atmosphere and to maximize student-student and student-teacher interactions, the teaching process was conducted in a different special classroom in out-of-school hours. The reason for conducting the study in a special classroom was that the classroom had a U-shape seating design which allowed placing a large table for the activities and displaying the works (number sentences) on the board. In the present study, this understanding dominated the teaching process, and the students were expected to do various mathematical tasks individually or via group interactions. Following this, the way the students structured the concepts were observed via the discussions regarding the tasks.

Table 1. *Preliminary and Last Interview*

Question		Preliminary Interview	Last Interview
1		Meaning of the symbol of "="	
2 True/false number sentences	a.	9 + 7 = 7 + 9	8 + 5 = 5 + 8
	b.	6 + 9 = 5 + 11	9 + 6 = 10 + 7
	c.	12 - (9 - 2) = (12 - 9) + 2	15 - (8 - 5) = (15 - 8) + 5
	d.	(5 x 4) x 7 = (7 x 4) x 5	(6 x 7) x 8 = (8 x 7) x 6
	e.	8 + (3 x 8) = (5 x 8) - 8	6 + (7 x 6) = (9 x 6) - 6
	f.	42 : 16 = 84 : 32	90 : 24 = 30 : 8
3 Open number sentences	a.	□ = 12 - 8	□ = 15 - 6
	b.	3 + 4 = □ + 5	5 + 9 = □ + 8
	c.	7 + 16 = □ + 15	9 + 19 = □ + 21
	d.	68 + 58 = 57 + 69 + □	64 + 56 = 65 + 55 + □
	e.	3 x 21 = 7 x □	4 x 18 = 9 x □
4 Open number sentences (involve adding)	a.	42 + 54 + □ = 56 + 45	63 + 25 + □ = 27 + 66
	b.	62 + 38 = 60 + □	98 + 26 = 96 + □
	c.	92 + □ = 95 + 85	80 + □ = 83 + 16
5 Open number sentences (involve subtracting)	a.	71 – 52 = 72 – □	62 – 45 = 63 – □
	b.	75 – 32 = 73 – 28 - □	48 – 17 = 46 – 16 - □
	c.	627 – 125 = 625 – 121 – □	514 – 236 = 512 – 232 – □
6 Open number sentences (involve adding, multiplying, dividing)	a.	5 x 8 = □ + □ + □ + □ + □	4 x 9 = □ + □ + □ + □
	b.	4 x 3 = □ x 3 + □ x 3	8 x 6 = (□ x 6) + (□ x 6)
	c.	(□ x 8) + (□ x 8)+ (□ x 8) = 9 x 8	(□ x 5) + (□ x 5)+ (□ x 5) = 12 x 5
	d.	(3 x 4) + □ = 3 x 7	(4 x 5) + □ = 4 x 9
	e.	15 : 5 = (10 : 5) + (5 : □)	30 : 10 = (20 : 10) + (10 : □)
	f.	□: 2= (4 : 2) + (4 : 2)	□ : 3= (6 : 3) + (6 : 3)
7 Open number sentences (involve adding, multiplying, subtracting)	a.	5 x 9 = 10 + 10 + 10 + 10 + 10 – □	7 x 9 = 10 + 10 + 10 + 10 + 10 + 10 + 10 – □
	b.	2 x 9 = (2 x 10) – □	4 x 9 = (4 x 10) – □
	c.	(8 x 9) + □ = 8 x 10	(7 x 9) + □ = 7 x 10
8 Open number sentences	a.	5 x (8 + 4) = (5 x □) + (5 x 4)	3 x (7 + 5) = (3 x □) + (3 x 5)
	b.	3 x (10 – 4) = (3 x □) – (Δ x 4)	4 x (9 – 5) = (4 x □) – (Δ x 5)
	c.	□ x (7+8) = (□ x7) + (□ x 8)	□ x (5 + 6) = (□ x 5) + (□ x 6)
9	a.	□ + □ = □ + □	□ + □ = □ + □
	b.	□ - □ = □ - □	□ - □ = □ - □
	c.	□ - □ = □ + □	□ + □ = □ - □

The teaching process was planned as six sessions; however, two more sessions (one after the third session and the other after the sixth session) were held to rehearse the subjects. Eventually, a total of eight sessions were organized. Considering the students' ages, the sessions lasted between 30 and 40 minutes, and when necessary, breaks were given during the sessions. The teaching process lasted eight weeks including one session for each.

Session 1: In the first session, the question of "What does equality mean to you?" was directed to the students to have them learn that equality refers to a relation. Teachers are suggested first to use Cuisenaire rods and coins to teach the relational meaning of the equal sign (Seo & Ginsburg, 2003). In this respect, Mikado rods, which were thought to serve the same function, were used in the first session. However, as in Cuisenaire rods, the colors of Mikado rods were not taken into account because the underlying intention was to allow the students to express the relationship between the addend and the sum using a simple language while doing different groupings with the rods. The activities designed with the Mikado rods were carried out within the context of addition, and each student was asked to form different groupings and to write down these groupings. The number sentences created were shown by the students in the table of t (Sample number sentences: $\square$ + 7= 20 + 8; 971+ 108=112+$\square$).

Session 2: In the second session, for the purpose of observing the extent to which the students made use of relational thinking in subtraction, 30 unit cubes were distributed to the students, and they were asked to remove as many cubes as they wished and to note down the number of the remaining cubes. Following this, the students were asked to determine the minuend, subtrahend and difference after each subtraction. In the end, the discussion part started. The students were asked about the quantities (minuend) for their different operations, and the changes were examined. When they said 'minuend', the changing quantities were discussed.

Figure 1. Scenes from the second session

Session 3: In the third session, 24 unit cubes were distributed. The students were asked to put the cubes in groups with 3 in each, and the way they would state the operation as a number sentence was discussed. In this activity, the students were expected to write down number sentences like "3+3+3+3+3+3+3+3=24", "3x8=24" and "8x3=(2x3)+(6x3)". Following this, they were asked to re-group these groups and to write down them as number sentences. In the second lesson of the third session, play-money banknotes were given to the students. 20 1TLs, 4 5TLs, 2 10TLs and 1 20TLs were distributed to the students equally making 20TLs in total for each student. Following this, the students were asked about the relations between these banknotes. They were expected to write down the qualities in number sentences, to show them in t table and to find out the relation between the multiplicators.

Session 4: The question of "Is it important when the places of the numbers are changed was directed to the students to let them interrogate whether addition and subtraction have the commutative property or not. In the second part, true/false number sentences which included the associative property were given to the students. They were allocated enough time to examine these number sentences, and the question of "Do you need to do calculation was directed to them to let them interrogate which sentences were false and why they were false and to allow them make related generalizations (Sample number sentence: 6 – 5 = 5 – 6).

Session 5: In the fifth session, the students were asked how to place the pieces of chocolate in a parcel. Based on the students' responses, several trials were done using a parcel brought into class, and the students were allowed to see that there were boxes in the parcel and pieces of chocolate in boxes. They were asked how to calculate the number of the chocolate pieces, and they were shown the pieces of chocolate in boxes. They were asked how to calculate the number of the chocolate pieces to seek for various other responses. Following this, they were asked to think about a sample question (*There are 10 biscuits in a package of biscuits. 20 packages of biscuits are put in a box. And one parcel houses 25 boxes. Accordingly, can you write down the statement that will show the number of biscuits in one parcel?).* Based on the students' responses, the students studied further with unit cubes. First, a rectangular prism with its top open was taken, and the students were asked how to place the unit cubes. In this phase, in order to help discover the distributive property, the unit cubes were selected in different colors. In addition, the students were given cubes and asked to put three cubes at the bottom and to examine the change in the number of the cubes (Sample number sentences: (8 x 4)+ __ = 8 x 6; 9 x 3 x 2 = 6 x 3 x __).

Figure 2. Scenes from the fifth session

Session 6: The division started with a problem situation given to the students (Example: *Triangular pyramids and square prisms are equal in number. The total face number of square prisms is 36. Then, find the total face number of the pyramids.).* The students were allowed to examine the total face numbers of the square prisms and triangular pyramids brought into class. They found the number of the prisms and discussed the equality of the numbers of the prisms and pyramids, and they were expected to find the total face numbers of the pyramids. In line with the results they obtained, they discussed whether it was possible to write down the equation of "24:4=36:6", and the data were noted down in t table. The session was carried on with examples to develop relational thinking in the division (Sample number sentences: 60 : $\square$ = 20 : Δ; 10: (5:5) = (10:5) :5).

Data Analysis

The data were analyzed in two phases: constant analysis and backward analysis. Constant analysis constitutes the basis of teachers' natural or planned guidance so that they can improve students' learning. The important point in this analysis is that it allows the researcher to form and arrange the research model in line with students' knowledge, actions and tendencies (Simon, 2000). In the constant analysis process of the study, the researchers watched the recorded videos at the end of each lesson, discussed their observations and the results in detail and noted down important points. Since student development had priority, the lesson plans were revised in line with these observations, and the lessons were repeated if necessary. In the backward analysis, both the clinical interviews and the teaching process were examined, and the data were gathered under three main themes by two independent field experts. The first theme, the operational process based on relational thinking, was divided in two sub-themes: *use of fundamental arithmetic properties of operations* and *use of the relation between numbers*. Use of fundamental arithmetic properties refers to students' thinking about commutative, associative and distributive properties of operations. Use of relations between numbers, the second sub-theme of the operational process involving relational thinking, requires considering the numbers and operations in both sides of an equation and seeking for a relation between the numbers given in a number sentence.

It was found that the students' responses to some of the questions could be said to involve relational thinking because these responses did not fully reflect the properties of relational thinking. In this respect, the students' explanations regarding their responses were thought to belong to the theme of operational process, which could be regarded as introduction to relational thinking and which was made up of two sub-themes: *Explaining the relation after finding the unknown* and *Pre-Relational Thinking.* In this theme, the first sub-theme was explaining the relation after finding the unknown, which included students' evaluation of the results they had found and their explanation of the relation between the numbers based on the result. The students found the result via result-oriented thinking; on the other hand, they were able to explain the relation between the operations after finding the result. This situation was found important in terms of directing students towards relational thinking. The second sub-theme of the introductory operational process to relational thinking was the stage of pre-relational thinking. According to this sub-theme, the students did not start with result-oriented thinking for the given number sentences, and they searched for the relations between numbers and operations; however, they eventually favored doing calculations as they were not completely sure about the relations.

As for the students' responses and explanations which did not involve any relational thinking skill were examined under the theme of *result-oriented operational process*. Result-oriented operational process means finding the result by doing calculation without seeking for any relations between the numbers given in the equations. The researchers reached an agreement on which theme the students' responses belonged to.

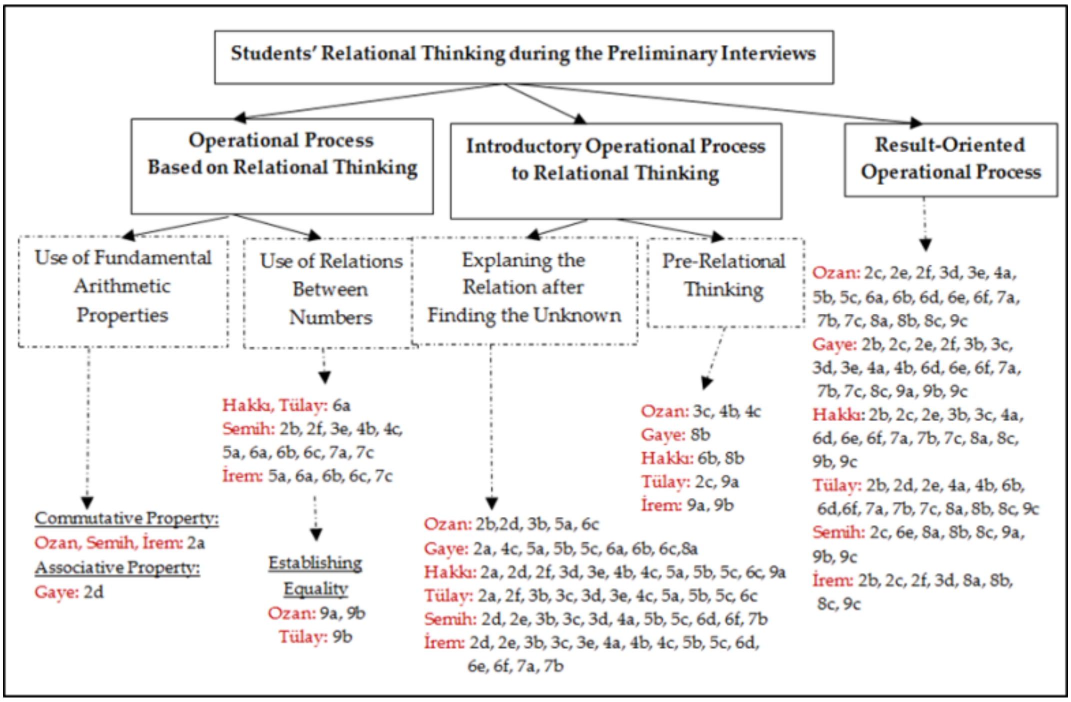

Figure 3. Students' relational thinking during the preliminary interviews

Findings

The findings obtained are presented under two headings: Findings related to the preliminary interview and those related to the last interview.

Findings Related to the Preliminary Interviews

The students were first shown the equal sign and were asked for their thoughts about the meaning of the sign. All the students reported that they recognized the equal sign, and they referred to this sign as "equality of two numbers" and "finding the result of an operation". However, it was seen that the students had difficulty stating different meanings of the equal sign and that they generally used it to find the result of an operation. To illustrate, one of the students explained the meaning of equal sign as "a sign used before the answer".

The students' ways of thinking during the preliminary interviews were examined under three headings: Operational process based on relational thinking, operational process as_an introduction to relational thinking, and result-oriented operational process. Figure 3 presents which process the students favored for which question. It is seen that the students mostly provided responses to the questions in the preliminary interviews in line with the result-oriented operational process. Table 2 demonstrates the results regarding the students' ways of thinking for 31 number sentences with respect to their levels of academic achievement.

Table 2. *Total Number of Number Sentences for the Students' Thinking Processes in Terms of Their Levels of Academic Achievement*

		Operational Process Based on Relational Thinking		***Introductory Operational Process to Relational Thinking***		***Result-Oriented Operational Process***
		U.F.A.P	U.R.B. N	E.R. A.F.U	Pre-R. T.	
Low level	*Ozan*	1	2	5	3	20
	Gaye	1	-	9	1	20
Middle Level	*Hakkı*	-	1	12	2	16
	Tülay	-	2	11	2	16
High level	*Semih*	1	11	11	-	8
	İrem	1	5	15	2	8

As can be seen in Table 2, the responses given by Ozan and Gaye, who were two of the students with low level of academic achievement, revealed that these students thought more on result basis when compared to other students. It was seen that Gaye found the question difficult when she failed to find the result and that she even thought it was impossible to write down the correct numbers in the boxes without any calculation. Similarly,

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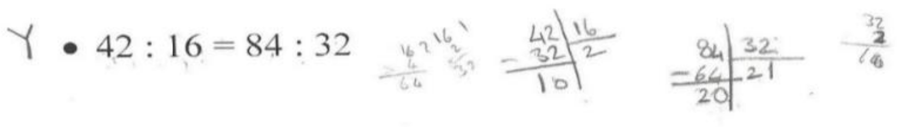
Semih, another student, preferred doing calculation rather than dividing the dividend (15) into pieces in the sentence of "15:5=(10:5)+(5: □)" [6e] and said one could not find the result without doing any calculation. To give an example for the result-oriented operational process, Gaye was unable to find the result of the division in the number sentence of 2f and decided that the number sentence was wrong. In the process, the student focused directly on division instead of searching for a relation between the numbers and failed to progress in division.



Teacher : *Now, what you did was to divide 42 by 16.*

Gaye : *If we multiply 16 by 2, it makes 32, but 10 is smaller than 16, well.*

Teacher : *You found 10 as the remainder. Yes, you say 16 is bigger than 10, you are right.*

Gaye : *I really don't know how to go on.* In addition, the result-oriented thinking process in another number sentence of "5 x 9=10+10+10+10+10–□" [7a] can also be given as an example for Gaye.

Gaye : *It could be 5.*

Teacher : *Why?*

Gaye : *Because the answer is 45. Well, this is 10, 20, 30, 40, 50. When we subtract 5 from 50, the result is 45.I calculated it.*

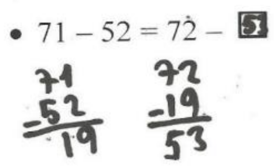
Teacher : *I see. There are many 10s here, right? There are 5 10s. Do you think you could have found the correct number for the box without doing any calculation? I mean without adding 10, 10 and 10?*

Gaye : *Actually, I think I wouldn't.*

During the preliminary interviews, Semih, one of the students, thought the most difficult question was "□x (7+8)= (□ x7) + (□x 8)" [8c] and said the reason was that there were a number of boxes in the number sentence. As a matter of fact, the reason why Semih experienced difficulty in in such questions was that he was unable to establish a relation between the numbers and operations and that he directly tried to find the result. This question was one of those which involved doing operation and which did not require the students to establish any relation. The fact that all the students focused on the result while answering this question, which was a good example for the distributive property of multiplication over addition, indicated that they did not have any knowledge about the distributive property.

One of the students, Gaye, reported that the most difficult questions were the last ones and stated the reason as follows "*I was unable to find the exact number, so I failed to find an exact number and thus wrote down the same number in the blanks*". The main reason here could be said to be the students' intention to find the result without establishing any relation between operations and numbers in the number sentence. This situation is supported especially by all the students' result-oriented thinking for the question of "□–□=□ +□". In addition, it was seen that all the students, except for Tülay, had difficulty in the number sentence of "12-(9-2)=(12-9)+2" [2c] and thought on result-oriented basis. The preliminary interviews revealed that the number sentences dominated by result-oriented thinking were those requiring multiplication and division, those requiring use of distributive property of multiplication over addition or subtraction, and those requiring operations in parentheses.

Figure 3 presents the second theme showing that the students intensively used the introductory operational process to relational thinking. As can be seen in Table 3, the students with moderate and high levels of achievement were able to recognize and explain the relations in relation to the results of the number sentences which especially included addition and subtraction. In addition, the fact that the teacher asked for each question whether it was necessary to do calculations to find the result helped the students seek for a relation between the numbers and operations. For example, one of the students, Ozan, tried to establish a relation regarding the number sentence of "71–52 = 72–□" [5a] based on his previous answers to other number sentences, yet he was not sure about the relation he established. It was seen that Ozan focused on the relation between the numbers based on the result he found after doing the calculations) and explained the difference.



Teacher : *What do you think?*

Ozan : *I subtract 52 from 71 and then subtract the result from 72.*

Teacher : *Then, you say you will do calculation again to find the result for the box. Well, could you find it without doing any calculation?*

Ozan : *Without any calculation? 71, here, the number increases 1-point, and in 52, probably, here, it increases 1-point again.*

Teacher : *But, you are not sure again.*

Ozan : *Well, yes, I am not sure. Actually, I really think that we need to do calculation.*

Teacher : *You think we have to calculate it. Then, you may calculate it if you wish.*

Ozan : *19. Then, if we subtract 52 from 71, it makes 19. So, when we subtract another number from 72, it must be 19. Then, to find it, we subtract 19 from 72, and it makes 53! This means the number is 53. Before finding the result for this box, I said there were a 1-point increase from 2 and a 1-point increase from 71 to 72. So, if we increase 52 by 1 point, I thought the result will probably be the answer. Then, it is possible to find the result just by increasing it by 1 point and without doing any calculation.*

Regarding the sub-theme of pre-relational thinking applied by a limited number of students from different academic achievement levels, Hakkı's thinking process for the number sentence of "3 x (10–4) = (3 x □)–(Δ x 4)" [8b] could be given as an example. Hakkı wanted to remove one of the operations in parentheses on the right side of the equation in the number sentence and wrote down 0 for the triangle. In this way, he wanted to establish a relation thinking that both sides of the equation would be multiplied by 3. However, he did a calculation as he was not sure of the process, and he failed to focus on the multiples.

It was seen during the preliminary interviews that they provided responses to some of the number sentences, though few in number, as appropriate to the operational process based on relational thinking. This theme, which examined how the students established relations between numbers and how they used and explained the properties of operations, was divided into two sub-themes: use of *fundamental arithmetic properties of operations* and *use of relations between numbers*. The preliminary interviews also revealed that some of the students (Ozan, Salih and İrem) found the number sentence of "9+7=7+9" [2a] correct by referring to the commutative property without doing any calculation, while some of the students (Tülay, Hakkı, Gaye) said it involved the commutative property after doing the calculation. A similar process could also be said to be true for the associative property. Gaye was the only student who said the number sentence of (5x4)x7=5x(4x7) was correct depending on the associative property without doing any calculation. Therefore, fundamental arithmetic properties were used by four students only for two questions (2a, 2d), in which the commutative and associative properties were clear enough to understand. The only question solved by four students via relational thinking was the number sentence of 5x8=□+□+□+□+□ [6a], in which the multiplication-addition relation was most obvious. In other questions, which did not require establishing equality_only the students with high levels of academic achievement were able to produce solutions based on the relation between numbers. For instance, İrem, one of the students, focused on the common multiples for the numbers in the sentence of "(8 x 9)+ □= 8x10"[7c] and found the number for the box by establishing a correct relation.

İrem : *It must be 8 because when you subtract 8x9 from 8x10, it makes 8. There is only one fold between them.*

Teacher : *Did you first think about the folds?*

İrem : *Yes, I did.*

The most different question regarding the use of a relation between numbers was the one requiring establishment of an equality which included open number sentences. Among the students, only Ozan and Tülay solved the equations of "□+□=□+□" [9a] and "□–□=□–□" [9b] by focusing on the difference between the numbers via relational thinking and by realizing that in subtraction, there will be a similar increase or decrease in both minuend and subtrahend.

Findings Regarding the Last Interviews

During the last interviews, the students were first asked to state the meaning of the equal sign again. It was seen that all the students stated the correct meaning the equal sign. The students pointed out that the equal sign had a meaning of balance rather than a meaning of result. For instance, one of the students, Ozan, used such definitions of the equal sign as "*teeter-totter, equality, balance, equal sides*". The students' responses during the last interviews were examined with respect to *operational process involving relational thinking, operational process as an introduction to relational thinking* and *result-oriented operational process*. Figure 4 presents the thinking processes followed by the students for each question.

As can be seen in Figure 4, the most important finding was that for all the tasks, almost all the students provided responses appropriate to the operational process involving relational thinking.

In this respect, it could be stated that all the students' relational thinking skills were developed. It was found that in this development process, there was a mutual interaction between the development of fundamental arithmetic concepts and the development of relational thinking.

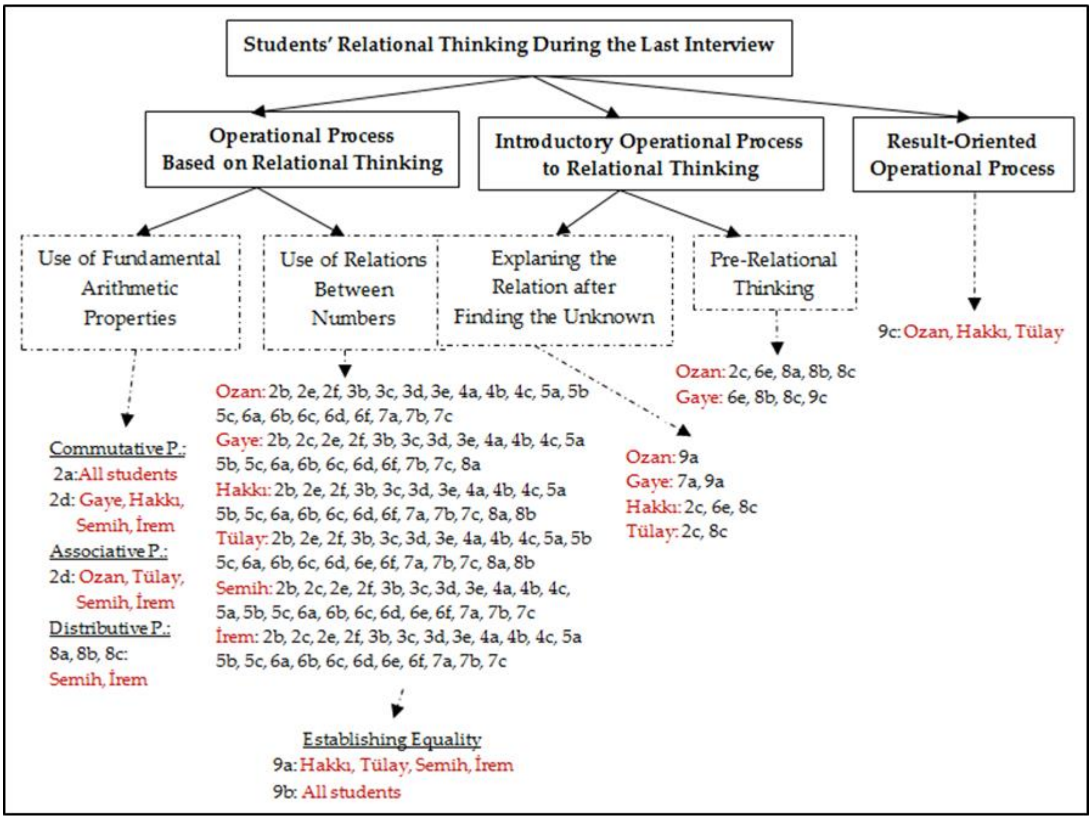


Figure 4. Students' Relational Thinking during the Last Interviews

It was also seen that the students developed their understanding of such concepts related to arithmetical operations as addend, sum, minuend, subtrahend, difference, multiplicator, product, dividend, divisor and quotient and that they used the relations between these concepts effectively. Table 3 presents the results regarding the students' ways of thinking for 31 numbers sentences with respect to their levels of academic achievement.

As can be seen in Table 3, the students did calculations based on relational thinking for the given number sentences. Especially within the scope of fundamental arithmetic properties, all the students proved the correctness of the number sentences via relational thinking without doing any calculations for two true/false number sentences (2a and 2d) which required use of commutative and associative properties.

For the 6th and 8th questions, which involved the distributive property, it was seen that the students with high levels of academic achievement were able to recognize the distribute property and found the result based on the property without doing any calculation. For example, Semih, one of the students, recognized the distributive property for the number to be written down for the box in the number sentence of "3x(7+5)=(3x☐)+(3x5)" [8a] and marked 3 as the common multiple.

Table 3. *Number of Total Number Sentences for the Students' Thinking Processes with Respect to Their Levels of Academic Achievement*

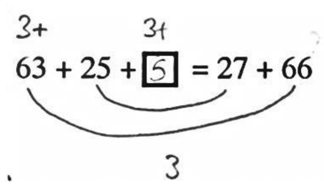
		Operational Process Based on Relational Thinking		***Introductory Operational Process to Relational Thinking***		***Result-Oriented Operational Process***
		U.F.A.P	U.R.B.N	E.R.A.F.U	Pre-R. T.	
Low Level	*Ozan*	2	22	1	5	1
	Gaye	2	23	2	4	-
Middle Level	*Hakkı*	2	25	3	-	1
	Tülay	2	26	2	-	1
High Level	*Semih*	5	26	-	-	-
	İrem	5	26	-	-	-

Semih : *Distributive... it was the distributive property. Here, 7 and 5 are added and multiplied by 3. And, here, it is multiplied separately.*

Teacher : *Multiplied separately...*

Semih : *Thus, they gave 5, and because the remainder is 7, I wrote down 7 here.*

During the last interviews, as can be seen in both Figure 4 and Table 3, in all the number sentences, in which only one single operation (addition, subtraction, multiplication and division) is given or multiplication and addition are given together, it was found that the students were able to establish a relation between the numbers via relational thinking. For instance, one of the students, Tülay, pointed out that there must be a similar decrease or increase in both sides of an equation by repeatedly saying "we have to establish a balance here, then...":



Tülay : *Here, it increased by 3 (66), and for the balance, we have to increase here by 3, (25) ... 25, 27...and we add 3 and 2. It makes 5.*

Teacher : *Why is it 5?*

Tülay : *There is a 3-point increase from 63 to 66. To establish the balance, I have to increase it by 3. It makes 2 from 25 to 27. If I add 3 and 2, it makes 5.*

Hakkı, who reported to have difficulty in division and though on result-oriented basis during the preliminary interviews, was found to do relational thinking during last interviews. It was seen that he recognized the common multiple in both sides of the equation in the true/false number sentence of "90:24=30:8" [2f] and explained the relation between the dividends and divisors as follows: "*I made use of the folds; 90 is three-fold of 30, and 24 is three-fold of 8. Therefore, it is equal.*"

To illustrate the use of a relation between numbers, the thinking process followed for the number sentence of "4x18=9x□" [3e] by Gaye, one of the students who had a low level of academic achievement yet demonstrated a striking development in relational thinking, could be given as an example. In this process, the student realized her mistake and explained the relation she established between the operations of division and multiplication in an equation also mentioning the rule she formed:

Gaye : *Teacher, it will be 2 here.*

Teacher : *Why?*

Gaye : *18 is divided by 2, and it makes 9. We will divide 4 by 2, and it makes 2.*

Teacher : *You will divide 4 by 2. Well, are you sure?*

Gaye : *No.*

Teacher : *Why not?*

Gaye : *Teacher, I did it wrong. When we divide 18 by 2, it makes 9, and 4 has to be multiplied by 2.*

Teacher : *Why?*

Gaye : *Teacher, in addition and multiplication, one side increases, and the other decreases. Here, if one side is divided, then the other will be multiplied.*

Teacher : *Very good. If one side decreases ... then you say there is such a case in addition and multiplication .... How do you know that; I mean how did you say this? I haven't told you such a thing before, but it is a good answer.*

Gaye : *Well, I have thought so via what you have taught us.*

Another example for the relation established by the students between the numbers and operations was the thinking process carried by Semih for the number sentence of "62–45=63–□" [5a]. The student clearly stated the relation between the minuend, subtrahend and difference for the subtractions in the given equation:

Teacher *: Did you write 46? How did you know that?*

Semih *: This (63) is 1 point higher than this one (62), and this (box) must be 1 point higher than that one (45).*

Teacher *: Why did you increase one side instead of decreasing?*

Semih *: Well, if in subtraction, the subtrahend increases, the difference will decrease (...) and when the minuend increases, the difference will decrease. This is the reason.*

The number sentence of "15-(8-5)=(15-8)+5" [2c], for which all the students did result-oriented thinking during the preliminary interviews, was solved via relational thinking by three students [Gaye, İrem, Semih] during the last interviews. İrem's focus on the subtrahend and difference in both sides of the equation can be given as an example:

İrem : *Teacher, this* [15 - (8 - 5) = (15 - 8) + 5] *is correct because, teacher, here* [on the left side of the equation], *8 is not subtracted directly, subtracting 5 from 8, and then subtracting 3 from 15. But, here* [on the right side of the equation], *subtracting 8, I mean 8 is higher, and 5 is added for the difference.*

It was seen in the last interviews that students with moderate and low levels of academic achievement had difficulty in some open number sentences (2c, 6e, 8a, 8b and 8c) and that they did not do result-oriented thinking in any of these questions. It was also found that the students who used the introductory operational processes to relational thinking failed to recognize the distributive property in some of these questions and that they did not understand why there were more than two boxes in some of these questions. This situation was supported by the fact that all the students with moderate and low levels of academic achievement had difficulty in the open number sentence of "□x(5+6)=(□x5)+(□x6)" [8c]. For instance, Ozan said only 11 could be written in the boxes, while Gaye realized that more than one number could be written in the box but failed to write

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down the numbers as she was not sure. Although both students identified the relation, they provided responses appropriate to pre-relational thinking by doing calculation. It was also seen that Hakkı and Tülay first did calculation and then placed the numbers based on their calculation.

The only question which is considered to be difficult for students within the scope of relational thinking and for which the students with a moderate level of academic achievement did calculations on result-oriented basis was the equation of "$\square + \square = \square - \square$" [9c]". Only İrem and Semih did relational thinking for this question and established relations between the numbers. For a good example, Semih made use of folds while establishing the equation:

$$\boxed{36} + \boxed{6} = \boxed{48} - \boxed{6}$$

***Teacher** : How did you think about it? Did you think about the result?*

***Semih :** No, I used the folds.*

***Teacher :** How?*

***Semih :** Here [36], it is 6 folds of 6, and I added one more fold, and it made 7 folds. Here [48], I thought it was 8 folds, and I subtracted 1 fold.*

Semih and İrem, who did relational thinking in all the last interviews, conducted the solution process quite fast, and in the process, it was seen that they developed their self-confidence. To illustrate, İrem noted in her diary at the end of the interview that "*I no longer do calculations at all. I always seek for relations between numbers, and I really like it*."

Conclusion, Discussion and Suggestions

According to a number of students and even to a number of adults, arithmetic refers to calculation of numbers and symbols based on a certain rule without establishing any relations (Carpenter et al., 2003). Such a situation constitutes an obstacle that prevents individuals from internalizing the properties and meaning of arithmetic operations, from establishing relations and even from producing in-depth mathematical thoughts. In order to avoid this obstacle and to develop students' mathematical reasoning skills regarding fundamental operations and properties, open and true/false number sentences are, as mentioned by Carpenter et al. (2003), considered to be an effective tool. In this way, students can give meaning to arithmetic operations and reflect this into properties of operations in a way to establish a sub-structure for the development of algebraic thinking. The most important result obtained in the present study, which focused on how to establish this sub-structure and on how to develop current thoughts, was that the students first responded to the open and true/false number sentences based on the result-oriented process during the preliminary interviews and that they then responded to these open and true/false number sentences based on the relations between numbers and operations in the last interviews. As in a study conducted by Napaphun (2012), secondary school students' working with open number sentences helps viewing the given equations_as a whole and allows them to use their relational thinking skills effectively. Therefore, it could be stated that the teaching process carried out in the study led to an important development of the students' relational thinking skills. This overall finding could be said to be consistent with the results of other studies which demonstrated that a teaching process based on the relations between open and true/false number sentences develops elementary school students' relational thinking skills (Carpenter et al., 2003; Koehler, 2004; Molina, Castro & Ambrose, 2005; Molina &Ambrose, 2008).

First, the present study investigated the meaning of the equal sign, which is thought to be the key to relational thinking. Parallel to the results of other studies which demonstrated that students have limited knowledge about the meaning of the equal sign (Behr et al., 1980; Kieran, 1981; Falkner et al., 1999; Molina et al., 2008; Saenz-Ludlow & Walgamuth, 1998; Yaman et al., 2003), it was found in the present study during the preliminary interviews that the students did not regard the equal sign as a symbol showing a relation but as "finding a result or doing something" and "a symbol of operation or a symbol-syntactic indicator used before the answer" (Warren, 2006). As a reflection of this result, it was seen that the students mostly provided answers to the open and true/false number sentences as appropriate to the result-oriented thinking processes and that they avoided establishing or failed to establish a relation between the numbers and operations for the equality. In addition, it was a striking finding that there were students who thought they would not be able to find the numbers to be written down in related boxes without doing any calculation. The reason for this situation is thought to be the fact that students learn arithmetic on result-oriented basis and that they focus on calculations rather than on relations between numbers and operations. This result is consistent with those of other studies which reported that students have difficulty understanding mathematical structures and relations (Warren, 2004) and that students tend to do calculations (Kieran, 2004).

In the study, the preliminary interviews revealed that the students mostly thought on result-oriented basis for the number sentences which involved multiplication and division and for those which required use of the distributive property of multiplication over addition or subtraction. The preliminary interviews also demonstrated that all the students had difficulty dealing with division, compared the divisions on each side of the equation via calculation and provided wrong answers. Considering the fact that the difficulty level of number sentences is important for students to recognize the relations between numbers in an equation (Carpenter & Levi, 2000), it is natural that number sentences involving multiplication and division are more difficult to deal with when compared to addition and subtraction. One possible reason for this could be the fact that students have difficulty in dealing with number sentences involving operations in both sides of an equation or that they are not accustomed to such number sentences. In related literature, there are several research results which report that number sentences in elementary school mathematics course books are mostly given as operations-equation-answer and that the number of examples supporting the use of relational thinking in number sentences is limited (Köse & Tanışlı, 2011).

Although most of the students with moderate and high levels of achievement thought on result-oriented basis for the number sentences involving addition and subtraction during the preliminary interviews, it was seen that they were able to explain the relations for the results they had found. This thinking process, which could be regarded as introduction to relational thinking, was thought to be important since the students managed to recognize the relations between the numbers and operations. Accordingly, this thinking process was regarded as a transitional phase in which the students started to see the relations and the fundamental arithmetic properties. The most important factor influential on this situation was the teacher's interrogative approach during the preliminary interviews. This approach is supported by the fact that the teacher encouraged the students to establish relations between the numbers and operations for the given number sentences. Thanks to this support, two students with a high level of achievement were able to establish a relation without doing any calculation especially for the number sentences requiring a fold relation (6a, 6b, 6c, 7c). In addition, it was another striking result that three of the students managed to recognize the commutative property among the fundamental arithmetic properties and one student was able to recognize the associative property. Considering the fact that the commutative property for addition is taught in elementary school 1st grade course of mathematics and that the associative property for multiplication is taught in elementary school 3rd grade course of mathematics, this situation indicates that students fail to internalize these fundamental arithmetic properties. Moreover, in relation to the questions involving the distributive property, it was found that the students were able to solve only the number sentence involving the relation between multiplication and addition based on relational thinking and that they experienced difficulty in all the other number sentences. Koehler (2004) points out that for relational thinking, number sentences involving the multiplication and addition relation could be regarded as transition to use of the distributive property in complex number sentences. Thus, for secondary school students, the distributive property could be a starting point for relational thinking.

In addition, it was found that the students who failed to say the relational meaning of the equal sign during the preliminary interviews were able to give correct meaning to this sign during the last interviews thanks to the teaching process conducted in the study and that they considered the sign to convey the meaning of 'balance' rather than 'result'. As a reflection of this result, almost all the students thought on relational basis regarding the given number sentences during the last interviews. This result is parallel to other research results which demonstrate that the fifth-grade class has a vital place for the development of the relational meaning of the equal sign (Rittle-Johnson et al., 2011). Another important finding was that the students thought on relational basis in all the number sentences involving a single operation (only addition, subtraction, multiplication or division) or a combination of multiplication and addition during the last interviews. Considering the fact that number sentences act as a window that reveals students' mathematical thoughts (Carpenter et al., 2003), the students' analysis of the open and true/false number sentences on the basis of relational thinking during the last interviews was another important finding. It was seen that in line with this analysis, the students developed their fundamental arithmetic concepts as well as such concepts related to arithmetic operations as addend, sum; minuend, subtrahend, difference; multiplicator, product; dividend, divisor and quotient and that they made effective use of the relations between these concepts even though they failed to provide formal expression of these relations. During the last interviews, the students, to solve the questions, focused on the common multiples of the numbers found in the equation and used the properties of operations to solve the questions which involved multiplication and division together. It was also seen that in division, the students compared the dividends and divisors on both sides of the equation and paid attention to the fact that the ratio between the dividends and the ratio between the divisors should be the same. In literature, there are several studies which include limited examples regarding subtraction and division and which demonstrate that students experience difficulty in operations in subtraction based on relational thinking (Kieran, 2007), while in the present study, it was found that the students with moderate and high levels of achievement had difficulty in some number sentences and that they did not think on result-oriented basis in an of the number sentences.

Based on the findings obtained in the study, it is thought that students' relational thinking skills can be developed at earlier ages. Elementary school teachers and mathematics teachers have great responsibilities for teaching not only the fundamental properties of operations but also the relations between numbers and operations via number sentences and in-class discussions. In this respect, especially elementary school teachers could be provided with in-service training in relation to how to develop relational thinking. An important point which mathematics teachers should remember is that students may lack knowledge about the meaning of the equal sign and about such fundamental concepts as arithmetic operations and related properties. Therefore, web-based professional development programs and in-service trainings could be organized to emphasize the role of relational thinking in providing a sub-structure for algebraic thinking.

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