

# Semiotic Representations in The Learning of Rational Numbers by 2<sup>nd</sup> Grade Portuguese Students

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## Abstract

The use of different registers to represent mathematical concepts enhances understanding. For example, rational numbers can assume pictorial, symbolic and natural language representations and this kind of change improves learning. Based on these assumptions, a teaching experiment for the learning of rational numbers by 2nd grade students was conducted, so as to allow for an understanding of how semiotic representations contribute to the learning of rational numbers, particularly with concern to unit fractions. Using a qualitative methodology and a content analysis of the students' written productions, the study shows a greater use of the pictorial representation register compared to the other types. Students' main difficulties in learning rational numbers are related to the pictorial representation of unit fractions and to an understanding of the concept of fraction itself. Some of these difficulties result from errors such as the misrepresentation of unit fractions in the case of the pictorial register, the association of the concept "half" with multiple unit fractions, the absence of the fraction bar when it comes to the symbolic register, the use of everyday terms to represent fractions when students rely on the natural language register, and the misrepresentation of rational numbers when the graphic register is used.

## Keywords:

Registers of Semiotic Representation; Mathematics; Learning Rational Numbers; Unit Fraction; Primary School.

## Introduction

Mathematics is a human activity and one of the oldest sciences, occupying an important place in the school curriculum. This school subject is quite different from most other subjects that students have to learn in school because its object of study is of an abstract nature (Davis & Hersh, 2020; Ponte et al., 2007). Mathematics is also seen as a language that allows students to develop an understanding and representation of daily life and "a tool that provides ways to solve problems" (Ponte et al., 2007, p. 2). Given the nature of mathematical elements, being unique with regard to teaching and learning, it implies using good and multiple representations of mathematical ideas (Duval,



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1995, 2003; Ponte et al., 2007; Ponte & Quaresma, 2012). Such a claim, which is widely accepted, is shown to be even more relevant for learning in the early years of schooling, since children's thinking, at this age, relies on the manipulation of concrete ideas (Canavarro & Pinto, 2012). An example of this is the teaching of rational numbers in the Portuguese early Primary school grades (1 to 4), which represents a great challenge for teachers, especially since the last curriculum changes determine that the learning of fractions, among other topics, should start earlier, in the earlier grades (Ministério da Educação e Ciência [MEC], 2013).

Lamon (2007) considers that fractions are, among all the topics that make up the curriculum and require more time for development and acquisition/learning, the most difficult to teach, the most complex (mathematically speaking), but also the most challenging and essential for the learning process. This complexity is also highlighted by some other authors (Fernandes, 2013; Kieren, 1992; Silva et al., 2014). In an attempt to provide an explanation for this complexity, Silva et al. (2014) point out that rational numbers are "difficult for students to understand because of the multiplicity of representation registers and associated meanings" (p. 1487).

Written representations "of mathematical ideas are an essential part of mathematical learning and production" (National Council of Teachers of Mathematics [NCTM], 2007, p. 75). Students can use and connect the different representations of a given mathematical concept to externalise their thoughts. According to NCTM (2007), it is important to challenge students to represent their mathematical ideas in ways that really mean something to them, even if those representations are somewhat unconventional at first. By using instructional strategies that encourage associations between different types of representations, students develop their understanding of concepts, think and communicate with others using mathematical language (Duval, 1995, 2003, 2017; Guerreiro et al., 2015).

The theory of semiotic representations (Duval, 1993, 1995, 2003, 2017) includes different types of representations that facilitate student learning and allow them to choose how to represent their ideas. These representations give students the opportunity to record, reflect, and store the learnings they will need in the future (Woleck, 2001). The distinction between the different representations and mathematical objects becomes fundamental, but this distinction is also one of the main difficulties the learning process involves (Duval, 2017). Considering these difficulties, the teacher must adapt the tasks and materials (s)he proposes (Guerreiro et al., 2015).

The complexity of learning rational numbers at such an early age and the role that semiotic representations play in learning them is widely recognised, so in this study we intend to understand how semiotic representations are used to learn rational numbers, more specifically fractions, in a 2nd grade class of Portuguese students. To achieve this goal, we set the following research questions: Which semiotic representational registers do students use most when learning non-negative rational numbers? What difficulties do students perceive and what mistakes do they make when learning rational numbers? By answering these questions, we hope to contribute to the knowledge of semiotic representations in the topic of fractions by students in the early years of schooling.

### ***Semiotic Representations***

Representations play a fundamental role in the learning of mathematics and are essential for an understanding of mathematical concepts (Duval, 1995; NCTM, 2007). This additional importance of representations in Mathematics, from Duval's perspective (1995) occurs because "mathematical objects are not directly accessible to immediate perception or immediate intuitive experience like the so-called "real" or "physical" objects. Representative forms of meaning are therefore necessary (p. 268)".

Representations allow students to think mathematically, to express their ideas, and at the same time they are instruments that students use to communicate those thoughts to others (Duval, 1995; NCTM, 2007). Woleck (2001) points out that "representations are not static products. On the contrary, they allow us to capture the process of building a mathematical concept or mathematical relationships" (p. 215). Noting down and reflecting on representations tends to promote the recovery of the thought processes that students use in the activities they carry out, allowing them to "articulate, clarify, justify and communicate their reasoning to others" (Woleck, 2001, p. 215).

Goldin and Stheingol (2001) consider two distinct groups of representations: external and internal representations. According to these authors, an external representation may include a representational system of mathematical symbols, such as the base-ten number system or the formal algebraic notation, but also a representational system involving concrete manipulable materials. Internal representations "include constructs of personal symbolisation that are used to assign a given meaning to mathematical notations, the students' natural language, their visual images and, more importantly, their relationship to mathematics" (Goldin & Stheingol, 2001, p. 2).

Duval (1995) suggests four different groups of representations: external, internal, conscious, and non-conscious. From this author's perspective, external representations are "closely linked to a state of development and control of a semiotic system" (p. 25), and are produced to translate ideas or concepts, through tables, diagrams, graphs, models, and symbols. For Dreyfus (2002), these representations, written or spoken, are essential to make mathematical communication possible between people. Duval (1995) suggests that conscious representations are "those that have an intentional character and fulfil an objectivation purpose" (p. 24). For the author, objectivation corresponds to the subject's own discovery of what (s)he had not suspected until then, even if (s) he had already had access to such information." (p. 24). Semiotic representations are external and conscious, and "allow for a 'vision of the object' upon perception of different types of stimuli (points, lines, characters, sounds...) conducive to different sorts of significance" (Duval, 1995, p. 27). Figures, schematics, graphs, and symbolic expressions are examples of semiotic representations that can be used in mathematics.

The theory of registers of semiotic representation (TRSR) was developed by Duval in an attempt to understand how knowledge is acquired in mathematics through the specificities of representations. Semiotics is the study of signs that carry meaning and significance that can be identified by human beings. According to Duval (1995), "the notion of semiotic representation entails (...) the consideration of different semiotic systems and a cognitive operation that will be capable of converting representations from one semiotic system into another" (p. 17). According to this theory, mathematical objects can be displayed through semiotic representations, which are defined as "productions consisting of the use of signs that belong to a given system of representations, which has its own restrictions of meaning and operation" (Duval, 1993, p. 39). It follows that a semiotic system represents a chance to manifest and interpret signs, each one carrying its own meaning.

In mathematics, there is a wide variety of semiotic representation registers. Duval (2003) suggests the existence of four types: natural language, algebraic and numerical writing systems, geometric figures, and graphs. In mathematical activity, the same object can be represented through different registers of semiotic representation, which means that the object is different from its representation. Each representation provides different information about the object represented, hence the importance of using different representation registers. The diversity of registers is important in that it becomes "a necessary condition to prevent mathematical objects from being mistaken

for their representations and to ensure that each one is recognizable" (Duval, 1993, p. 40). According to this author, a semiotic system can be a register of semiotic representation if it allows for the three fundamental cognitive activities linked to semiosis: the creation of an identifiable representation; treatment; and conversion. The treatment of a representation consists of a transformation carried out in the same register (Duval, 1995). The conversion of a representation is the transformation of a representation into a representation of another register, retaining all or part of the content of the initial representation. Unlike treatment, it mobilises different registers of representation and presents an external transformation of the source register. This activity "is a complex transformation, much more complex than the operation of treatment, because any change in register requires the recognition of the object shown in the two representations whose content is often very different" (Duval, 1995, p. 112). If students are unable to anticipate a conversion to be made, or to recognise an object in two distinct representations, it will be very hard for them to solve a given task suggested by the teacher.

### *Semiotic Representations Of Rational Numbers*

Representations are the basis of mathematics, unlike other areas of knowledge where it is possible to observe facts without representing them (Canavarró & Pinto, 2012; Duval, 2017; Ponte et al., 2007). For example, representing a number allows "assigning a designation to that given number and students have to understand that a number can have several designations" (Ponte & Quaresma, 2012, p. 40). These designations are assigned through the different possible representations of the same number. In the case of rational numbers, they can be represented by a fraction, decimal number, percentage, diagram, among others, that must be considered in the classroom (Morais et al., 2014). A decimal number results from the fraction  $\frac{a}{10^n}$ , where  $a$  is an integer and  $n$  is a natural number.

If we follow the Portuguese curriculum for mathematics in Primary Education and assume that  $a$  and  $n$  are natural numbers, we obtain a set of positive decimal numbers (MEC, 2013). When students fully understand the different representations of rational numbers, they are able to develop their thinking skills (Canavarró & Pinto, 2012; Ponte & Quaresma, 2012), which reflects itself in the way they are able to "communicate their reasoning to others" (NCTM, 2007, p. 240).

Mathematics learning depends on several factors (e.g., cognitive, social, cultural, and contextual) and on the different actors involved in the educational process, among which the role of the teacher is

to be emphasized. Currently, recommendations for mathematics education indicate that students should be able to master the use of the different representations of rational numbers and not just memorize concepts (Barnett-Clarke et al., 2010; Canavarró & Pinto, 2012; Hodges et al., 2008; Kara & Incikabi, 2018; Özsoy, 2018; Scaptura et al., 2007). In order to enable familiarization with rational numbers, we need to represent them. Rational numbers can presume a symbolic representation, e.g.,  $\frac{2}{5}$ , a decimal representation, 0.4, or a percentage figure, 40%, or other kinds of possible registers of representations, such as an expression in natural language or a pictorial (also called iconic) representation (Canavarró & Pinto, 2012; Brandl et al., 2016; Özsoy, 2018).

Contact with fraction representation is very common and can lead to misunderstanding in the context of learning rational numbers, as students may believe that they are rational numbers and not a mere form of representation. Fractions are the first register of representation of rational numbers that students come into contact with and therefore can be defined as "two-sided symbols, a particular way of writing numbers:  $a/b$ . This particular meaning of the word fraction refers to a notational system, a symbol, two integers separated by a bar" (Lamon, 2007, p. 635). This is how the concept of fraction is understood, and there is evidence that it is not always properly explained to students and that they only remember the way fractions are written and not what they really stand for. In order to understand rational numbers, students need to understand that "all rational numbers can be written in the form of fractions; that there are numbers written in the form of fractions that are not rational numbers, like  $\pi/2$  for instance; and that each fraction does not correspond to a different rational number, like  $\frac{2}{3}$ ,  $\frac{6}{9}$ , for example" (Lamon, 2007, p. 635).

Students who are able to use different representations of rational numbers have mastered the concept of rational numbers and are aware that they are much more than just a simple way of representing rational numbers (Kara & Incikabi, 2018). Therefore, the teacher should encourage students to use rational number representations flexibly to promote the acquisition and development of knowledge (Lemonidis & Piliandis, 2020).

### ***Rational Numbers In Mathematics Curricula***

In mathematics curricula, the development of the concept of number, its meaning and its operations and properties in a given number field is considered a central learning objective (MEC, 2013; Ponte et al., 2007). From preschool to the final term of high school, students develop their knowledge of numbers, namely their concept of numbers and how they are "represented by objects, digits, or straight lines; how

they relate to each other; how they are (...) in systems with certain structures and properties; and how they should be used to solve problems" (NCTM, 2007, p. 34). In the first years of school, students learn about different types of numbers and become capable of distinguishing, for example, which numbers are even, odd, prime, connected or fractions (Brocardo & Carrillo, 2019; Canavarró & Pinto, 2012; Guerreiro et al., 2018).

In Portugal, the first numbers that students learn about in 1st grade are the natural numbers, as part of subtopics such as "correspondences one to one and comparison between the number of elements of two sets; counting up to twenty objects; the empty set and the number zero; counting natural numbers up to 100, and, counting on and back" (MEC, 2013, p. 7). Natural numbers, while addressed in preschool education through informal operations, require more systematic work to give meaning to these numbers so that students can competently solve computations and problems involving computations, for example.

In 2nd grade, the concept of numbers broadens with the introduction of non-negative rational numbers, among other new concepts, with subtopics like "fractions  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{10}, \frac{1}{100}$  and  $\frac{1}{1000}$  as measures of lengths and other quantities; and, representation of natural numbers and fractions  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$  and  $\frac{1}{5}$  on a number line" (MEC, 2013, p. 9).

To introduce non-negative rational numbers, the current recommendations for the teaching of mathematics suggest that it should be done using everyday situations and common fractions that students use in their natural language (Guerreiro et al., 2018; Hunt et al., 2016). In 2nd grade, students should be able to understand when it is possible to divide a unit into equal parts, and this approach is more important than focusing on how fractions are represented (NCTM, 2007).

Students should also be able to identify parts of a unit divided into equal parts, such as three quarters of a sheet of paper folded into four equal parts and understand that 'quarters' means four equal parts of a unit (NCTM, 2007). The integer division is also introduced in 2nd grade so that students can understand that fractions are associated with division and associate the terms 'half', 'third part', 'fourth part' and 'fifth part' with the respective fractions (MEC, 2013).

In 3rd grade, the set of non-negative rational numbers includes three additional contents: non-negative rational numbers; adding and subtracting non-negative rational numbers represented by fractions; and decimal representation of non-negative rational numbers. The teaching of negative rational numbers includes the representation of several

fractions as a measure of length and other quantities, the representation of fractions on a number line, equivalent fractions, comparing and ordering fractions with the same numerator or the same denominator, among other subtopics, broadening the knowledge students had acquired in the previous year (MEC, 2013). Gradually, the concepts related to rational numbers interact with their procedural components as we introduce mathematical operations with non-negative rational numbers, namely addition and subtraction, and fractions with the same denominator. In 4th grade, the notion of non-negative rational numbers is further enlarged as students learn how to obtain equivalent fractions multiplying the top and bottom by the same factor and understand what it requires to simplify fractions (MEC, 2003). Operations get more complex as they learn how to multiply and divide non-negative rational numbers.

The way rational numbers are organised in school curricula allows for the existence of several different meanings, such as: part-whole/measurement, quotient, operator and ratios and rates (Oliveira, 2014). Part-whole/measurement refers to the division of a whole into equal parts or to the representation of a fraction as a single point on a number line. The quotient is based on the division of two natural numbers. It is like dividing 10 loaves of bread among two persons ( $10 \div 2$ ).

The fraction can be used as an operator when there is a transformation, something that acts on something and modifies it, for example, by multiplying  $\frac{2}{3}$  by something, we can first multiply by 2 and then divide by 3.

Ratio is used to compare two similar quantities, but "when a ratio represents the comparison of quantities of a different nature and can be conceived as the description of a phenomenon common to other situations, this comparison is considered a rate" (Oliveira, 2014, p. 70).

Rational numbers are introduced through fractions. This concept is not considered a priority in the pre-school to 2nd grade curriculum; however informal experiences at these levels will help lay down and develop mathematical bases that will be relevant to further learning (NCTM, 2007). Early contact with the unit fractions, like 'one half' and 'one third' for instance, allows students to better understand the meaning of the fractions, which will then make it easier for them to use fractions to solve tasks and problems. The development of the various meanings of the fractions, over the different school grades, provides students with a higher level of resources that will make their work much easier.

## Method

This study aims to understand the contribution of semiotic representations to the learning of rational numbers, and specifically of unit fractions in the 2nd grade. Bearing in mind this particular objective, one of the authors carried out, during the last year of his master's degree, a teaching experience based on the use of mathematical tasks that called for the use of different semiotic representations of topics such as 'Introduction to unit fractions', 'Unit fractions: Part-whole', 'Dividing the unit: Number line', 'Unit fractions: Part-whole on a number line and using figures'. This teaching experience took place over four lessons (those that are curricularly defined) and included nine mathematical tasks, three of which are analysed in this paper. The different steps followed to solve each of the tasks include: the introduction, whose aim was to clarify the content of the task and to make sure that the students had understood what was expected; the exploration of the task that focuses on the students' resolutions; and the discussion of those resolutions.

The class in which this teaching experience was carried out consisted of 26 Portuguese students, 14 boys and 12 girls, aged between 7 and 8, each with a different learning pace. Although some students showed difficulties, most of them seemed to like mathematics.

Given the nature of the goal outlined, a qualitative and interpretative approach was adopted in order to understand the mathematical tools students resorted to, for solving the proposed classroom tasks (Bogdan & Biklen, 1994; Erickson, 1986). For that purpose, a wide range of data was collected, using various resources. For this paper, the data collected from the documental analysis reflects student performance in the proposed tasks.

Data analysis focused on the analysis of the content of student responses, and was translated, in a first moment, by the frequency distribution of the types of correct answers, partially correct answers, incorrect answers and of no answer situations. Subsequently, this analysis of the answers given by the students to each of the tasks focuses on the types of register of semiotic representation they used: RLN: register of natural language; RP: pictorial register; RS: symbolic register; RG: graphical register.

## Results

When we introduced the topic 'Unit Fractions', students explored the division of the unit into equal parts, a topic already studied with natural numbers, and provided a representation of the situations covered in the tasks selected for the study of this topic. The first task the students had to perform, in pairs, was the following:



**Task 1**

In a 2nd grade math class, João learned some geometric shapes and, when he got home, he decided to make those shapes using cardboard sheets of different colors. In the meantime, his friend Rui arrived and ended up destroying the geometric figures, cutting them into tiny pieces. João was very upset and threw the figures to the ground and scattered them all over the floor. Rui told him that he had divided the figures into equal parts to show him what he had learned at school.

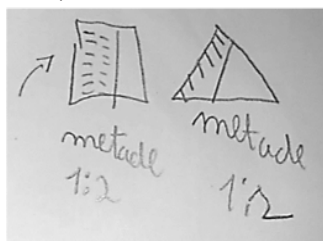
1. Once they put the geometric figures back together, Rui said "I divided the triangle and the square into two equal parts". How can we represent each of these parts?

2. Rui explained what he did with the triangle and the square. He also explained to his friend that he divided the rectangle into three equal parts, the circle into four equal parts and the pentagon into five equal parts to show him the numbers he had learned in his math class. What kind of numbers are these?

The division of the triangle and of the square into two geometrically equal parts highlighted the symbolic representation of this activity. During the discussion involving this representation, the students found that the results obtained are not always exact and, in such cases, the division can be represented by a fraction. Of all the parts of the figures the students encountered, halves were those they understood better, eventually because they remind students of common everyday situations. The clarification of the notion of half/halves of a geometric figure was seen as an occasion for focusing on its representation in the different registers. The representation of each part of the figures (square and triangle) was done correctly by four pairs of students. These representations are expressed in pictorial registers, symbolic registers and registers of natural language, as can be observed in the answer provided by pair P8 (Figure 1).

**Figure 1**

Correct answer provided by pair P8 to question 1 of task 1 (means half)

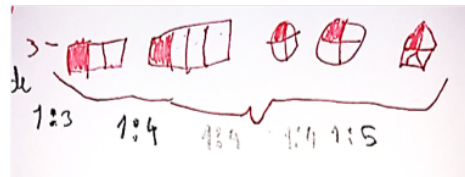


The other four pairs of students did not answer Question 1, which shows that they did not understand what they were supposed to do.

Question 2 of Task 1 proved to be fundamental in ascertaining if the students understood the concept of unit fraction. Two pairs of students answered correctly and were able to identify the numbers that represent the equal parts of a rectangle, a circle, or of a pentagon, using symbolic register and pictorial register, as shown in the resolution of pair P5 (Figure 2).

**Figure 2**

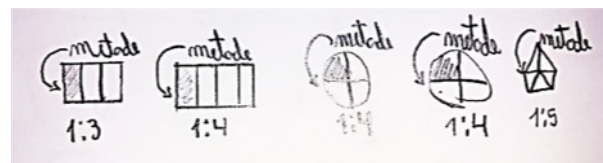
Correct answer provided by pair P5 to Question 2 of Task 1



As for the answers that were considered partially correct, two pairs of students correctly performed the pictorial and symbolic representation of each of the situations considered but did not establish the right connection between these representations and the register of natural language they would need to describe the numbers they had identified, as shown by the resolution of pair P1 (Figure 3).

**Figure 3**

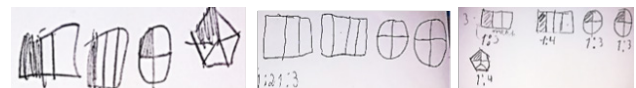
Partially correct answer given by pair P1 to Question 2 of Task 1



The other three pairs of students whose answer was partially correct provided the correct division of the figures, but one of the pairs did not symbolically represent any of the numbers that explain the division; the other pair only displayed part of these numbers (1/2 and 1/3) and did not represent the pentagon divided into five geometrically equal parts; and the third pair did not symbolically represent the division of some of the figures (Figure 4).

**Figure 4**

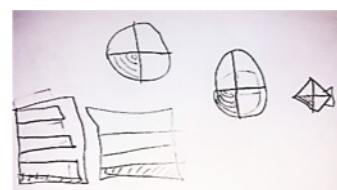
Partially correct answers provided, respectively, by pairs P4, P3 and P7 to Question 2 of Task 1



In the incorrect answer given to Question 3, the pair P8 incorrectly represented the division of the figures and did not indicate the numbers corresponding to this division (Figure 5).

**Figure 5**

Incorrect answer provided by pair P8 to Question 2 of Task 1



**Table 1**  
Frequency of the types of register used by the students in Task 1

Question	Types of answers											
	C				PC				I			
	RG	RS	RLN	RP	RG	RS	RLN	RP	RG	RS	RLN	RP
1.	0	4	4	4	0	0	0	0	0	0	0	0
2.	0	2	0	2	0	4	0	5	0	0	0	1
Total	0	6	8	12	0	4	2	7	0	0	0	1

Note: RG: Graphical register; RS: symbolic register; RLN: register of natural language; RP: pictorial register.

The division of geometric figures, in this grade, does not usually demand that their construction be entirely thorough, especially when it involves figures with several sides like the pentagon. Therefore, students, when challenged to divide geometric figures, often end up by putting at risk the meaning assigned to each of the parts of the unity.


In addition to analysing the students' answers to the questions from Task 1, it is important to identify the type of registers they have used in each of these questions (Table 1).

It appears that the pictorial record was the most commonly used for solving the questions in Task 1. It was used 12 times by the pairs of students who answered correctly, 7 times by the pairs of students who provided partially correct answers and once by a pair of students whose answer was incorrect. These results show that students find it easier to express their ideas using drawings than through other registers. This situation is quite natural since this ability is greatly developed in young students as soon as they enter pre-school.

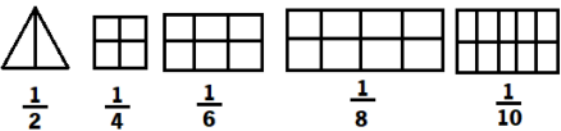
Then, students solved Task 2 to deepen previous learning.

**Task 2**


1. Write down the fractions that correspond to the colored part in each of the situations.



2. Color the figures according to the fractions suggested.



3. Indicate, in each situation, the colored part using two different representations.



Students completed this task individually so that we could understand whether they had understood the topic 'Unit Fractions'. The results are shown in Table 2.

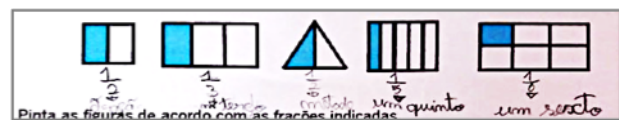
**Table 2**  
Frequency of the type of answers provided for the questions of Task 2

Question	Types of answers			
	C	PC	I	NR
1.	7	3	2	4
2.	8	7	0	1
3.	6	3	3	4

Note: C: correct; PC: partially correct; I: incorrect; NR: no answer provided.

In the first question, seven students (43.75%) indicated the correct fraction that corresponds to the coloured part of each of the figures using symbolic register. However, one of these students answered using two types of representation register: symbolic register and register of natural language, as we can see in the answer provided by student A20 (Figure 6).

**Figure 6**  
Correct answer provided by Student A20 to Question 1 of Task 2



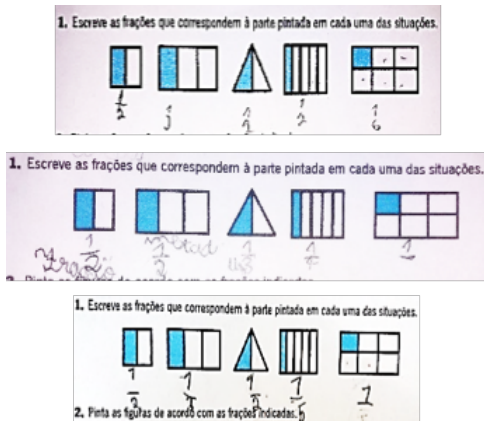
The use of both registers reveals that the student has not only understood what he was taught about unit fractions, but also that he feels confident enough to use different registers, showing that the same mathematical "object" can be represented in various ways. As he converts the symbolic register into natural language, the student uses, in the first figure, the term "fraction" referring to  $\frac{1}{2}$ , but in the third figure he writes the correct term to describe that same fraction, which shows a certain lack of critical thinking.

The answers given by three of the students (18.75%) are only partially correct because student A8 does not indicate all the corresponding fractions; student A26 answers correctly only to three of the five fractions, forgetting to place the fraction bar between the numerator and the denominator; and student A11 uses

symbolic representations and natural language to represent the coloured parts of the figures using fractions that do not correspond to the figure (Figure 7).

**Figure 7**

Partially correct answers given, respectively, by students A8, A26 and A11 to Question 1 of Task 2

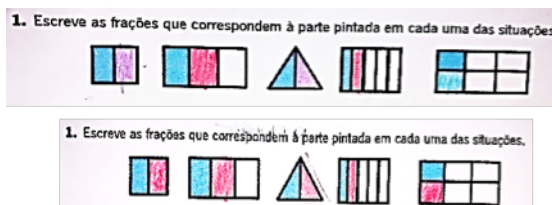


The analysis of student A26's partially correct answer shows that he correctly represents the fraction in the first figure but omits the fraction bar that separates the numerator from the denominator in the other figures. The way student A11 presents his fractions tells us that he or she did not understand the topic addressed, since the only correct answer is the fraction that he or she uses for the first figure. The representation of the fraction  $\frac{1}{2}$  in natural language is only correct in the second figure, in which the student states that the corresponding fraction is  $\frac{1}{2}$ , which in the register of natural language is represented by 'half'.

The answers given by two of the students (12.5%) are incorrect because the students did not mention the corresponding fraction - the students coloured one of the parts on the figures instead of using the symbolic representation of the corresponding fraction. This indicates that they did not understand what they were asked to do (Figure 8).

**Figure 8**

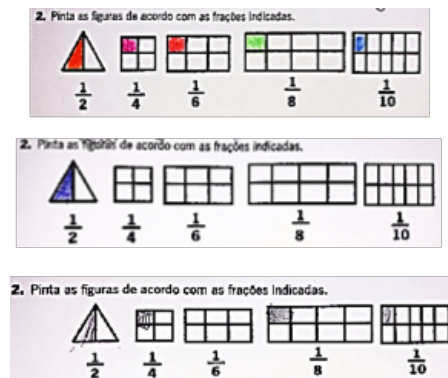
Incorrect answers given, respectively, by students A25 and A16 to Question 1 of Task 2



In Question 2, eight students (50%) answered correctly, painting only one part of each figure, as registered by student A18 (Figure 12). However, seven students (43.75%) provided answers that were only partially correct. They did not provide a full answer to the question, since they only coloured correctly a part of some of the figures, as can be seen in the answers given by students A10 and A3 (Figure 9).

**Figure 9**

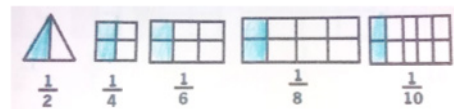
Correct answer given by student A18 and partially correct answers given, respectively, by students A10 and A3 to Question 2 of Task 2



The remaining five students whose answer was partially correct coloured two parts of each figure, only the first having been done correctly (Figure 10).

**Figure 10**

Partially correct answer given by student A20 to Question 2 of Task 2



Analysing the partially correct answer provided by student A20, it seems that he or she was able to establish a direct correspondence between the part coloured in the first figure and the others. The student did not consider the numerator of each fraction presented, though. The students who provided such an answer revealed that they did not understand what each unit fraction represents, failing to successfully carry out the conversion from the symbolic register into the pictorial register for each of the fractions.

Six students (37.5%) answered correctly to Question 3 of Task 2, but only one of them used two distinct registers of representation, the symbolic register, and the register of natural language. The latter was used by only one of the six students and the former was used by four of the six students, as expressed in the answers given by students A8, A4 and A11 (Figure 11).

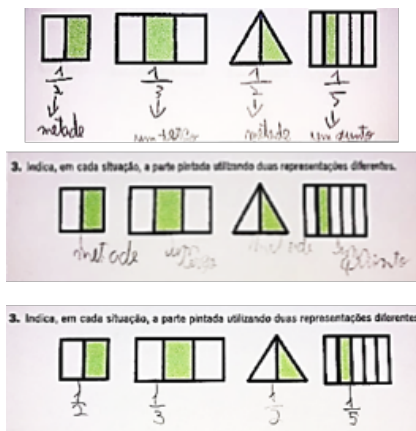
The partially correct answers were given by three students (18.75%), who only provided a correct representation of two fractions, as shown by the answer given by student A5 (Figure 12).

The incorrect answers given by three students (18.75%) show once again that the students did not understand what they had to do, as shown by the answer given by student A16 (Figure 13).



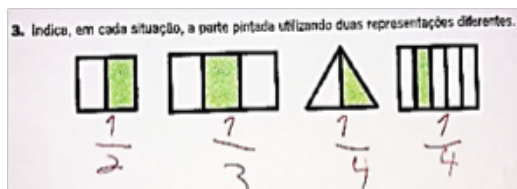
**Figure 11**

Correct answers provided, respectively, by students A18, A4 and A11 to Question 3 of Task 2



**Figure 12**

Partially correct answer given by student A5 to Question 3 of Task 2



**Figure 13**

Incorrect answer given by student A16 to Question 3 of Task 2



As for the type of register that students use in the answers given to the questions of Task 2, symbolic register is the most widely used by students who provided correct answers, while pictorial register is the most commonly used in the partially correct and incorrect answers (Table 3).

The symbolic register used in most of the correct answers made it possible for students to acquire

competences regarding the use of rational numbers and, more specifically, unit fractions. Through the symbolic representation of the operation of division, most students showed that they had understood what each fraction represented in the various situations. Using this type of register, the students were able to provide correct answers.

The pictorial register used by the students to answer Question 1 led to incorrect answers. They failed to perform what they were asked to do - which was to write down the fractions that corresponded to the coloured part of each figure - and failed to understand the instructions they were given. This type of register was more widely used to answer Question 2 and made it possible for students to obtain partially correct answers by answering correctly to a part of the situations presented.

The representation registers used in the questions that are part of the tasks can influence student responses, denying them the possibility of using different representations, as was the case in the aforementioned task. To understand this relationship, students were asked to solve Task 3 whose instructions used only the register of natural language. The students could use the representation register they wished to reply to what was required in the task.

**Task 3**

The math teacher gave the students a rectangular chocolate bar and asked them to divide it into as many parts as they wanted. Maria decided to divide the chocolate into two parts, João into three parts, Rita into four parts and Rui into five parts. If each one of them eats one part of his/her chocolate bar, who will eat the greater amount of chocolate? Explain how you reached that answer.

The analysis of the students' answers to this task, which they solved individually, showed that half of the class did not provide any kind of answer, whereas six students from the other half (37.5%) provided a correct answer, one of the students (6.25%) gave a partially correct answer and another student (6.25%) an incorrect answer (Table 4).

**Table 3**

Frequency of the types of register used by the students in Task 2

Question	Types of answers											
	C				PC				I			
	RG	RS	RLN	RP	RG	RS	RLN	RP	RG	RS	RLN	RP
1.	0	7	1	0	0	3	1	0	0	0	0	2
2.	0	0	0	8	0	0	0	7	0	0	0	0
3.	0	5	2	0	0	3	0	0	0	0	0	3
Total	0	12	3	8	0	6	1	7	0	0	0	5

Note: RG: graphical register; RS: symbolic register; RLN: register of natural language; RP: pictorial register.

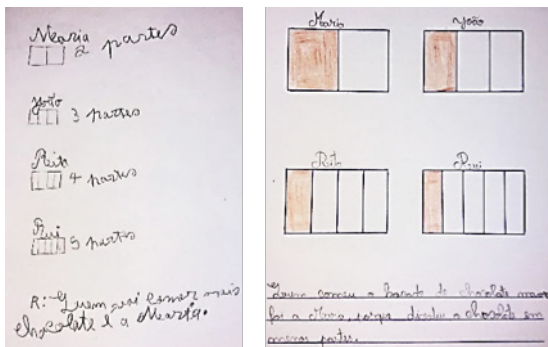
**Table 4**  
Frequency of the types of answers provided to Task 3

Task 3	Types of answers			
	C	PC	I	NR
Task 3	6	1	1	8

Note: C: correct; PC: partially correct; I: incorrect; NR: no answer provided.

In their responses, six students (37.5%) used a pictorial register to share their line of reasoning and answered that Maria was the one who ate the greater amount of chocolate, which was the correct answer. The register of natural language was used by four of these students to justify their answers, showing what they thought was the answer to this task, as shown by the answer given by students A5 and A20 (Figure 14).

**Figure 14**  
Correct answers given, respectively, by students A5 and A20 to Task 3

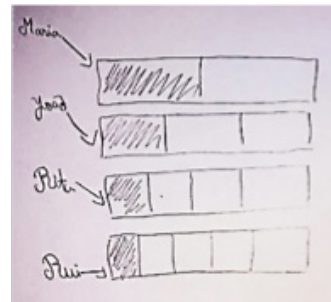


The pictorial register of the division of each chocolate bar allowed the students to realize how much each person would eat, making it easier for them to solve the task. The register of natural language completed the pictorial register and confirmed the identity of the person who had eaten more chocolate.

In the partially correct answer, one of the students (6.25%) pictorially represented the situation described in the problem but did not use the register of natural language to answer the question (Figure 15).

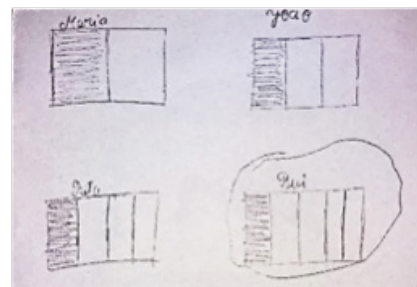
Student A7 answered the question using a pictorial register and this allowed him to see how much each person would eat. The final answer was not given though, which indicates that, in the student's understanding, the final answer is implicit in the pictorial representation he provided.

**Figure 15**  
Partially correct answer given by student A7 to Task 3



The incorrect answer given by one of the students (6.25%) used the pictorial register and the student considered the number of pieces that each person had obtained after the division, but experienced difficulties when he tried to compare fractions with the same numerator and different denominators in this register (Figure 16).

**Figure 16**  
Incorrect answer given by student A4 to Task 3



The difficulty that led to the incorrect answer given by student A4 is due to the quantity of pieces that each one obtained when the chocolate bar was divided and to the fact that he did not take into account that the size of each of the newly found pieces decreases as the number of chocolate pieces increases.

The analysis of the students' answers in terms of register shows that the pictorial register is the most widely used by the students regardless of the type of answer they provide (Table 5).

The use of the pictorial register allowed the students to visualize the concrete division of a rectangular figure, making it easier to understand how much each person would eat of a chocolate bar. The incorrect answer that used the pictorial register was due to the number

**Table 5**  
Frequency of the types of register used by the students in Task 3

Task 3	Type of answers											
	C				PC				I			
	RG	RS	RLN	RP	RG	RS	RLN	RP	RG	RS	RLN	RP
Task 3	0	0	4	6	0	0	0	1	0	0	0	1
Total	0	0	4	6	0	0	0	1	0	0	0	1

Note: RG: graphical register; RS: symbolic register; RLN: register of natural language; RP: pictorial register.

**Table 6**

Frequency of the types of register used by the students when they learn non-negative rational numbers.

Topic	Types of answers											
	C				PC				I			
	RG	RS	RLN	RP	RG	RS	RLN	RP	RG	RS	RLN	RP
Unit fractions: Introduction	0	18	15	26	0	10	3	15	0	0	0	7
Unit fractions: Part-whole	0	23	22	63	0	24	26	35	0	11	1	22
Unit Fractions: Part-whole on a number line and in figures	2	5	17	2	21	11	28	21	11	5	16	12
<b>Total</b>	<b>2</b>	<b>46</b>	<b>54</b>	<b>91</b>	<b>21</b>	<b>45</b>	<b>57</b>	<b>71</b>	<b>11</b>	<b>16</b>	<b>17</b>	<b>41</b>

Note: RG: graphical register; RS: symbolic register; RLN: register of natural language; RP: pictorial register.

of pieces into which the figure was divided, and the student failed to understand that each person would only eat one piece of chocolate, regardless of the division carried out.

The analysis and interpretation of the 2nd grade students' answers showed us how important it would be to summarize the types of registers of semiotic representation that were used by the pupils in learning about unit fractions (Table 6).

The analysis of Table 6 reveals that the most widely used register of semiotic representation was the pictorial register, in all kinds of responses. This register is constantly present in the learning process of students since Pre-school Education and this familiarity might have influenced its use in the proposed tasks. The symbolic registers and natural language, on the other hand, were used mainly to complement the students' responses, and were therefore used less frequently than the pictorial register. The graphical register was rarely used, which shows that students feel uncomfortable when they have to represent fractions on a number line. When the students used this register, they showed difficulties putting at risk their learning of rational numbers and, in this case, of unit fractions. It was difficult for students to make representations using this register because of the subdivision of the unit and because of the lack of additional information on the origin and on the unit itself.

The registers of representation chosen for solving tasks was, several times, influenced by the instructions received. However, when the instructions given did not suggest the type of register of representation that should be used, students tended to use the pictorial register. It was followed by the symbolic register and by the register of natural language.

Students found it difficult to carry out some of the tasks, namely: representing and dividing geometric figures with many sides; converting symbolic register representations into the register of natural language; distinguishing between the term 'fraction' and the symbol  $\frac{1}{2}$ ; interpreting the instructions given to carry

out the task; converting the colouring of geometric figures into rational numbers; and representing rational numbers using the graphical register. The work carried out with representations on a number line was not thorough enough for students to be prepared to use it correctly. It was therefore hard for them to complete the tasks involving graphical register.

The mistakes the students made while performing the tasks had to do with: identifying the coloured part in a geometric figure divided into equal parts; associating the concept of "half" with all the unit fractions; failure to use the fraction bar between the natural numbers that form it; using everyday terms to represent the fractions in natural language, for example, using terms such as 'one second' and 'one-third' to refer to the fractions  $\frac{1}{2}$  and  $\frac{1}{3}$ ; the confusion between the different registers of representation when it comes to carrying out a conversion; and the incorrect representation of the numbers 1, 2 and 3, that make students refer to the number 'two' as 'one half' in the register of natural language.

## Conclusions

This study shows that the pictorial register of semiotic representation is the register most widely used by 2nd graders when they present the line of reasoning that supports the resolution of the mathematical tasks used to teach them about 'rational numbers', more precisely about 'unit fractions'. The importance of pictorial representations in the learning of mathematics is highlighted by several authors, especially in the first years of schooling (Brocardo & Carrillo, 2019; Canavarro & Pinto, 2012; Guerreiro et al., 2018). In this regard, Canavarro and Pinto (2012) emphasize that "teachers need to use these representations to foster the presentation of conventional mathematical symbols and the writing of mathematical expressions, linking them to the corresponding iconic representations" (p. 76).

The mathematical tasks proposed in this study were fundamental to introduce students to the different registers of semiotic representation and were also

important to promote their understanding of rational numbers and help them distinguish mathematical objects from their representations (Duval, 2017). According to this author, mathematical understanding comes from distinguishing between mathematical objects and their representations, and to achieve such a goal, working with the different registers becomes increasingly relevant. These registers allow students to engage in mathematical work and through it they are able to build knowledge about the objects studied, since these objects do not have a tangible existence (Bonomi, 2015; Duval, 1993, 2017).

Involving the tasks given to them, students predominantly use pictorial registers, which is understandable, given their level of education (Canavarro & Pinto, 2012; Duval, 1993). However, they are also strongly influenced by the representations proposed in the instructions they receive. Notwithstanding the predominance of the pictorial register, the symbolic register and the register of natural language have been useful to improve and enlarge what has been previously taught and learnt. These are two registers of representation of rational numbers that are at an early stage of development and not yet consolidated enough to allow for a more effective use (Duval, 2003, 2017). Furthermore, students are still struggling to learn how to write in their own natural language and this has undeniably affected the register of mathematical representations.

Analysis of students' resolutions of the proposed tasks revealed that the process of learning about unit fractions is quite difficult. These difficulties are related to the understanding of the concept of unit fractions and their different forms of semiotic representation, a critical problem in mathematics education that several authors have pointed out (Brandl et al., 2016; Canavarro & Pinto, 2012; Duval, 2017; Goldin & Shteingold, 2001; Kara & Incikabi, 2018; Oliveira, 2016). Students' difficulties also relate to the division of geometric figures, the representation of fractions using the natural language register, the consideration of the part corresponding to the fraction of a figure, the change from one register to another, the representation of fractions using the graphical register and the interpretation of task instructions. These difficulties are evident because students make mistakes when responding, which must be discussed in order to overcome future difficulties, as pointed out by several authors (Kara & Incikabi, 2018; Canavarro & Pinto, 2012; Özsoy, 2018).

Finally, this study shows the existence of a conceptual development in the learning of rational numbers as a result of the use of the different registers of semiotic representation, since they help to modify the numerical field that normally limits the construction of student knowledge about mathematical objects.

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